1. **Optimization in One Variable**

Ref. J. Stewart, *Calculus* (or any 1st yr calculus textbook)

Review from *Calculus I* (e.g. MATH 100)

Want to find maximum or minimum value of a function (e.g. the "best" solution)

Function $y = f(x), x \in D$ - domain

- Absolute maximum value: $f(c) \geq f(x)$ for all $x \in D$ where $c$ belongs to $D$
- Local maximum value: $f(c) \geq f(x)$ for all $x$ near $c$ in $D$

( $D$ is all $x$ in an open interval intersected with $D$, containing $c$)

- Critical point (= critical number): a point $c$ in $D$ such that $f'(c) = 0$ or $f'(c)$ does not exist

- If $D = [a, b]$ is a closed interval, then the abs. max. value and the abs. min. value is attained at a critical point or an endpoint

- If a critical point is not an endpoint then the 1st Derivative Test or the 2nd Derivative Test can be used to determine if a local max or min. value is attained at $c$.

- If $D$ is not a closed interval, then sometimes $y = f(x)$ needs to be graphed to find abs. max or min. values.

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**Example 1:** Find the abs. max and min values of $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \leq x \leq 4$.

Note domain $D = [-\frac{1}{2}, 4]$ is a closed interval: easiest case.

Locate critical points, if any.

- $f'(x) = 3x^2 - 6x = 3x(x - 2)$ (factoring) very useful
- $f'(x) = 0 \iff x = 0$ or $x = 2$

To answer the question with least amount of work, just evaluate $f$ at each point and critical points, and list $y$-values

- $f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 + 1 > \frac{1}{8}$
- $f(0) = 1$
- $f(2) = (2)^3 - 3(2)^2 + 1 = -3$, $f(4) = (4)^3 - 3(4)^2 + 1 = 17$

Abs. max value

Abs. min value
However for more insight (and review) let us apply the 1st Derivative Test:

\[ f'(x) = 3x(x-2) \]

- If \(-\infty < x < 0\), \(f'(x) > 0\) \(f\) is increasing \(\rightarrow\) local max value \(f(0)\)
- If \(0 < x < 2\), \(f'(x) < 0\) \(f\) is decreasing \(\rightarrow\) local min value \(f(2)\)
- If \(2 < x < +\infty\), \(f'(x) > 0\) \(f\) is increasing

2nd Derivative Test:

\[ f''(x) = 6x - 6 = 6(x-1) \]

- If \(-\infty < x < 1\), \(f''(x) < 0\), \(f\) is concave downward \(\rightarrow\) \(f''(0) = -6\) \(\Rightarrow\) absolute max value \(f(0)\)
- If \(1 < x < +\infty\), \(f''(x) > 0\), \(f\) is concave upward \(\Rightarrow\) absolute min value \(f(2)\)

Graph of \(y = 3x^2 - 4\), \(-\frac{1}{2} \leq x \leq 4\) confirms earlier result.

\[
\begin{align*}
\text{Critical points:} & \quad (0,1) \text{ local max} \\
& \quad (4,17) \text{ absolute max} \\
& \quad (-\frac{1}{2},-\frac{1}{4}) \text{ absolute min} \\
\end{align*}
\]

Note: graphs of polynomials of low degree only have a few possible shapes.

Modeling (e.g., "word problems")

Mathematical modeling: take a "real-life" problem (e.g., science, engineering, economics...) you want to solve, translate it into a mathematical problem (called a model) and solve it. Analyze the model using mathematical methods (applied mathematics) to find a mathematical solution. Interpret this solution in terms of the real-life problem. Think about it: (test it, critique it, improve it...)

\[ y = 3x^2 - 4, \quad -\frac{1}{2} \leq x \leq 4 \]
Example: Optimal clutch size (e.g. David Lack)

What is the "best" number of eggs to lay (size of clutch)?

Considerations

1. More eggs, more babies hatching
2. More offspring, less resources per offspring, some die before reaching maturity

Want to maximize "fitness" = average * number of offspring that reach maturity, per parent.

* or, probably: think of repeating the "experiment" a large number of times, then taking the average for each experiment.

We expect fitness to be a decreasing function of the number of eggs laid, N. Why?

< Some discussion on "causal sense" >