\[ x_{n+1} = \frac{rx_n (1-x_n)}{f(x_n)} \]

Fixed points

\[ x^* = 0 \ (\text{all } r) \]
\[ x^* = 1 - \frac{r-1}{r} \quad (1 < r \leq 4) \]

+ Linearized stability

Fixed points of \( f^2(x) = f(f(x)) \) that are not already fixed points of \( f(x) \):

\[
\begin{align*}
p &= \frac{r+1 - \sqrt{(r-3)(r+1)}}{2r} \\
q &= \frac{r+1 + \sqrt{(r-3)(r+1)}}{2r}
\end{align*}
\]

\( 3 < r \leq 4 \)

\( f(f(p)) = p, \ f(f(q)) = q \)
\( f(p) = q, \ f(q) = p \)

\[ \begin{array}{c}
p \\
\bigcirc \!
\end{array} \]

\[ \begin{array}{c}
f \\
q
\end{array} \]
Linearized stability of $p, q$ as fixed points of $f^2(x)$:

$$
\mu = (f^2)'(p) = \frac{d}{dx} f(f(x)) \bigg|_{x=p} \\
= f'(f(p)) f'(p) \bigg|_{x=p} \quad \text{(Chain Rule)} \\
= f'(f(p)) f'(p) \\
= f'(q) f'(p) \\
= (r-2r q)(r-2rp) \\
= (r-(r+1)-\sqrt{(r-3)(r+1)})(r-(r+1)+\sqrt{(r-3)(r+1)}) \\
= (-1-\sqrt{(r-3)(r+1)})(-1+\sqrt{(r-3)(r+1)}) \\
= 1 - \frac{(r-3)(r+1)}{r^2-2r-3} \\
= 4 + 2r - r^2 \quad (3 < r \leq 4) \\
\text{If } r = 3, \quad \mu = 1 \\
\text{If } r > 3, \quad \frac{d\mu}{dr} = 2-2r < 0 \\
\mu \text{ decreases as } r \text{ increases}
$|\mu| < 1 \text{ i.e. } -1 < \mu < 1$, at least for $r > 3$ and $\nu$ sufficiently close to 3 and both $p, q$ are stable as fixed points of $f^3(\nu)$.

If $r$ increases further, $\mu = -1$ is possible

$\mu = 4 + 2r - r^2 = -1$

$\iff r^2 - 2r - 5 = 0$

$r = 1 \pm \sqrt{6} \quad (2 < \sqrt{6} < 3)$

$-2 < 1 - \sqrt{6} < -\frac{3}{2}$ this value is irrelevant but $3 < 1 + \sqrt{6} < 4$
i. If $3 < r < 1 + \sqrt{6} \approx 3.449$
then $p, q$ are stable

ii. If $r = 1 + \sqrt{6}$
then linearized stability fails

iii. If $1 + \sqrt{6} < r \leq 4$
then $p, q$ are unstable

Bifurcation diagram so far

What does it mean for $p, q$ to be stable/unstable as fixed points of $f^2$?

For $r = 3$, fixed points of $f(x) = 3x(1-x)$
are $x^* = 0, \ x^* = \frac{3}{3}, \ and \ \mu = f'(\frac{3}{3}) = -1$
(linearized stability fails)
Staircase/cubeweb diag. for $f$

$y = x$

slope of tangent is $\mu = -1$

$x^x = \frac{2}{3}$ is stable

Staircase/cubeweb diag. for $f^2$

$y = f(f(x)) = 9x(1-x)(1-3x+3x^2)$

Every iterate of $f^2$ corresponds to two itertes of $f$
\[ x_{n+2} = f(f(x_n)) \]

When \( 3 < r < 1 + \sqrt{6} \), fixed points of \( f \) are \( x^* = 0 \), \( x^* = 1 - \frac{1}{r} \), and \( \mu = f'(1-\frac{1}{r}) > -1 \).

Fixed points of \( f^2 \) are \( 0, 1 - \frac{1}{r}, p, q \).

Staircase/cobweb for \( f^2 \)

Every iterate here corresponds to two iterates of \( f \).
\[ \{ x_0, x_1, x_2, \ldots \} \] "converges" to \( \{ p, q \} \)

We say the 2-cycle \( \{ p, q \} \) is stable if

they are stable as fixed points of \( f^2(x) \)

To investigate what happens for \( 1 + \sqrt{6} < r \leq 4 \):

"Numerical" bifurcation diagrams

e.g. use Matlab, Maple, Excel, calculator

\[ x_{n+1} = r x_n (1 - x_n) \]

Fix \( r \)
Pick arbitrary \( x_0 \) (e.g., \( x_0 = 0.1 \)).

Compute \( x_n \), \( n = 1, 2, 3, \ldots \)
up to some large \( n \).

If \( x_n \) are converging to a stable fixed point \( x^* \) then \( x_n \approx x^* \) for large \( n \).

i.e. \( x^* \) is a "steady state" solution.

\[ x_n = \text{steady state} + \text{transient} \Rightarrow \text{transient} \to 0 \text{ as } n \to \infty \]

Discard the first \( N \) (e.g. \( N = 350 \)) points so transient \( \approx 0 \).

E.g. \( r = 0.5 \) (\( x^* = 0 \) is a stable fixed point)
\( x_0 = 0.1 \)

Then \( x_1 = 0.045 \)
\( x_2 = 0.0214875 \)
\[ \vdots \]
\( x_{351} = 0.0000000 \)
\( x_{352} = 0.0000000 \)

Note: there is roundoff error.
Actually \( x_{351} \approx 10^{-18} \).
Plot $n$ vs $x_n$

Plot $r$ vs $x$ only plot $x_{351}, x_{352}, \ldots, x_{500}$

Do systematically for small increments in $r$

Plots points close to $x^* = 1 - \frac{1}{r}$