1. Here is one more model of a fishery,

\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - \frac{M N}{N + A}, \quad N \geq 0, \]

where \(N(t) \geq 0\) is the number of individuals (fish) at time \(t\), and where \(r\) [time\(^{-1}\)], \(K\) [individuals], \(M\) [individuals\(\cdot\)time\(^{-1}\)] and \(A\) [individuals] are positive constants.

(a) Notice that the total rate that fish are taken out of the population (i.e. caught) \(\frac{MN}{N + A}\) saturates (approaches a limit from below as \(N \to \infty\)) as a function of \(N\). In what fisheries situation could this be a reasonable model?

(b) Give a fisheries interpretation of the parameters \(M\) and \(A\). What do they measure? What might affect their values in an actual fishery?

(c) Show that the system can be written in dimensionless form as

\[ \frac{dx}{d\tau} = x(1 - x) - \mu \frac{x}{x + a}, \quad x \geq 0 \]

for suitably defined (find their definitions) dimensionless quantities \(x, \tau, \mu, a\).

(d) Find all nonnegative equilibria \(x^*\). Show that the system can have one, two, or three nonnegative equilibria, depending on the values of positive \(\mu\) and \(a \neq 1\). Sketch phase portraits for the different values of positive \(\mu\) and \(a \neq 1\).

(e) For fixed \(i) 0 < a < 1 \text{ and } ii) a > 1\), treat \(\mu\) as the bifurcation parameter and draw the (global) bifurcation diagram in the \((x, \mu)\) plane (with \(\mu\) on the horizontal axis, \(x\) on the vertical axis), showing all the nonnegative equilibria and their stabilities for all positive \(\mu\). Classify any local bifurcations graphically.

(f) Interpret your results biologically with regards to how the value of the parameter \(M\) (or its nondimensional version \(\mu\)) affects the possibility of extinction. Could overfishing ever cause extinction (according to this model)?

2. A room containing 1000 m\(^3\) of air is initially free of carbon monoxide (CO). Then smoke containing 4% CO (by volume) enters the room at a rate of 0.1 m\(^3\)\(\cdot\)min\(^{-1}\), and the well-circulated mixture is allowed to leave the room at the same rate. At what time after the smoke first begins to enter the room does the concentration in the room reach the dangerous level of 0.012%?

3. Here is an improved model of the solid state laser considered in class. Let \(n(t) \geq 0\) be the number of photons in the device at time \(t\), and let \(N(t) \geq 0\) be the number of excited atoms. A more realistic model for the laser is the coupled system of differential equations,

\[ \frac{dn}{dt} = GnN - kn, \quad \frac{dN}{dt} = -GnN - fN + p, \]
where the gain coefficient $G$, the decay constant $k$ for loss of photons through the end mirrors and scattering, and the decay constant $f$ for loss of excited atoms through spontaneous emission are all positive constants, and the pump strength $p$ is a constant which may be positive, zero or negative. (All variables and constants have appropriate units.)

(a) We can convert this coupled system to a single equation, using an approximation that is reasonable in some situations. Suppose $N(t)$ changes on a much slower time scale than $n(t)$ does, so that $dN/dt$ is much smaller in magnitude than $dn/dt$ is. Then we can make a quasi-static (or adiabatic) approximation $dN/dt = 0$. Use this approximation to find $N$ in terms of $n$, then derive a first-order differential equation for $n(t)$.

(b) For the first-order differential equation for $n(t)$ that is the result of part (a), find all nonnegative equilibria $n^*$. Treating $p$ as a bifurcation parameter (with $G, k, f$ fixed) find (analytically) and classify (graphically) all local bifurcations.