The equation
\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - H \]
is a simple model of a fishery. In the absence of harvesting \((H = 0)\), the population \(N\) of fish grows according to the logistic equation. The effects of harvesting (fishing) are modeled by the term \(-H\), which says that the fish are caught or harvested at a constant rate \(H\) [individuals \(\cdot\) time\(^{-1}\)], independent of the number of fish, the weather, etc.

(a) Show that the system can be written in dimensionless form as
\[ \frac{dx}{d\tau} = x(1 - x) - h, \]
for suitably defined (find their definitions) dimensionless quantities \(x\), \(\tau\), \(h\).

(b) Plot the vector field for the different values of \(h\). Find all equilibria, determine the linearized stability of each equilibrium, and sketch phase portraits for different values of \(h\).

(c) Show that a bifurcation occurs at a certain critical value \(h_c\), and classify this bifurcation.

(d) Interpret your results biologically with regards to how the level of harvesting affects the possibility of extinction. Explain what could happen (according to this model) if fisheries managers thought the optimal rate of harvesting is \(h_c\) (consider what would happen if fishers even slightly exceeded their assigned rate of harvesting).

(e) Discuss limitations of this model (for example, are its predictions always reasonable?). In what situations might the model be useful? Discuss briefly how the model might be improved.

2. Here is another model of a fishery,
\[ \frac{dN}{dt} = rN \left(1 - \frac{N}{K}\right) - EN, \]
where \(E\) [time\(^{-1}\)] is a positive constant measuring the total effort made to harvest the given species of fish. You could think of it as proportional to the number of identical fishing boats allowed out on the body of water. The rate of harvesting is equal to the total effort times the number of fish in the body of water. (For example, if there are no fish, the rate of harvesting is zero, unlike in the previous model.)
(a) Show that the system can be written in dimensionless form as

\[ \frac{dx}{d\tau} = x(1 - x) - \varepsilon x, \]

for suitably defined (find their definitions) dimensionless quantities \( x, \tau, \varepsilon \).

(b) For \( 0 < \varepsilon < 1 \), find all equilibria, and determine the linearized stability of each equilibrium.

(c) A **sustainable yield** \( y \) is a rate at which fish can be harvested indefinitely. It is the product of the effort \( \varepsilon \) and any stable equilibrium population \( x^* \), i.e. \( y = \varepsilon x^* \). Find the dimensionless yield \( y \) as a function of the dimensionless effort \( \varepsilon \); plot the graph (known as the yield-effort curve) of this function.

(d) Determine \( \varepsilon \) so as to maximize \( y \) (with justification) and thereby find the **maximum sustainable yield** \( y_m \). What does this value correspond to in the original model (in terms of \( N, t, r, K, E \))?

(e) Explain what this model predicts will happen to the fish population if the actual effort of fishers is slightly above the critical value of \( \varepsilon = \varepsilon_m \) that gives the maximum sustainable yield \( y_m \).