1. $x' = x + y, \quad y' = 4x - 2y$

Write the system in vector form

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} =
\begin{pmatrix}
1 & 1 \\
4 & -2
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}.
$$

The characteristic equation is

$$
0 = \det \begin{pmatrix} 1 - r & 1 \\ 4 & -2 - r \end{pmatrix} = r^2 + r + 6 = (r - 2)(r + 3),
$$

so the eigenvalues are $r_1 = 2, r_2 = -3$.

For $r_1 = 2$, solve for $v^{(1)} = (v^{(1)}_1, v^{(1)}_2)^T$:

$$
\begin{pmatrix}
1 - 2 & 1 \\
4 & -2 - 2
\end{pmatrix} v^{(1)} =
\begin{pmatrix}
-1 & 1 \\
4 & -4
\end{pmatrix}
\begin{pmatrix}
v^{(1)}_1 \\
v^{(1)}_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

to get $v^{(1)}_1 = v^{(1)}_2$, or $v^{(1)}_1 = c_1, v^{(1)}_2 = c_1$ where $c_1$ is arbitrary. This can be written as

$$
v^{(1)} =
\begin{pmatrix}
c_1 \\
c_1
\end{pmatrix} = c_1
\begin{pmatrix}
1 \\
1
\end{pmatrix}.
$$

Choosing $c_1 = 1$ (for example) gives an eigenvector $v^{(1)} = (1 \ 1)^T$ (any nonzero constant multiple of this also will work).

For $r_2 = -3$, solve for $v_1 = v^{(2)}_1$ and $v_2 = v^{(2)}_2$:

$$
\begin{pmatrix}
1 - (-3) & 1 \\
4 & -2 - (-3)
\end{pmatrix} v^{(2)} =
\begin{pmatrix}
4 & 1 \\
4 & 1
\end{pmatrix}
\begin{pmatrix}
v^{(2)}_1 \\
v^{(2)}_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
$$

to get $v^{(2)}_2 = -4v^{(2)}_1$, or $v^{(2)}_1 = c_2, v^{(2)}_2 = -4c_2$ where $c_2$ is arbitrary. Choose $c_2 = 1$ an eigenvector $v^{(2)} = (1 \ -4)^T$ (or any nonzero constant multiple).

Since the eigenvalues are real and distinct, the general solution is

$$
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -4 \end{pmatrix}
$$

in vector form, or

$$
x(t) = c_1 e^{2t} + c_2 e^{-3t}, \quad y(t) = c_1 e^{2t} - 4c_2 e^{-3t}
$$

in component form, where $c_1$ and $c_2$ are arbitrary constants.

For the phase portrait, note that there is exponential growth along the eigendirection of $v^{(1)} = (1 \ 1)^T$ (if $c_2 = 0$) and exponential decay along the eigendirection of $v^{(2)} = (1 \ -4)^T$ (if $c_1 = 0$). Plot these special cases first, to organize the rest of the solution curves (plots...
of \((x(t), y(t)), \ -\infty < t < \infty\) in the phase portrait, which are hyperbolas in general. The origin is an unstable saddle point. See Figure 1.

2. (a) \(x'_1 = 5x_1 - x_2, \quad x'_2 = 3x_1 + x_2\)

Write the system in vector form

\[
\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

The characteristic equation is

\[
0 = \begin{vmatrix} 5 - r & -1 \\ 3 & 1 - r \end{vmatrix} = r^2 - 6r + 8 = (r - 4)(r - 2),
\]

so the eigenvalues are \(r_1 = 4, \ r_2 = 2\).

For \(r_1 = 4\) we solve

\[
\begin{pmatrix} 5 - 4 & -1 \\ 3 & 1 - 4 \end{pmatrix} \mathbf{v}^{(1)} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

to get an eigenvector \(\mathbf{v}^{(1)} = (v_1 \quad v_2)^T = (1 \quad 1)^T\). For \(r_2 = 2\) we solve

\[
\begin{pmatrix} 5 - 2 & -1 \\ 3 & 1 - 2 \end{pmatrix} \mathbf{v}^{(1)} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
to get an eigenvector $\mathbf{v}^{(2)} = (1 \ 3)^T$. Since the eigenvalues are real and distinct, the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

in vector form, or

$$x_1(t) = c_1 e^{4t} + c_2 e^{2t}, \quad x_2(t) = c_1 e^{4t} + 3c_2 e^{2t}$$

in component form, where $c_1$ and $c_2$ are arbitrary constants.

(b) $x_1' = -3x_1 + 2x_2, \quad x_2' = -x_1 - x_2$.

i) First, write the system in vector form

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$ 

The characteristic equation is

$$0 = \begin{vmatrix} -3 - r & 2 \\ -1 & -1 - r \end{vmatrix} = r^2 + 4r + 5$$

and the eigenvalues are complex:

$$r_{1,2} = -2 \pm j$$

(i.e., $r_1 = \lambda + j\mu$, $r_2 = \lambda - j\mu$, where $\lambda = -2$, $\mu = 1$).

ii) For the complex eigenvalue $r_1 = -2 + j$ we find a complex eigenvector $\mathbf{v}^{(1)} = (v_1 \ v_2)^T$, where

$$\begin{pmatrix} -3 - (-2 + j) \\ -1 \end{pmatrix} v_1 + 2 \begin{pmatrix} 2 \\ -1 - (-2 + j) \end{pmatrix} v_2 = 0,$$

or

$$(-1 - j)v_1 + 2v_2 = 0,$$

$$-v_1 + (1 - j)v_2 = 0.$$ 

The first equation can be seen as $(1 + j)$ times the second equation, so we can use either equation to find a relation between $v_1$ and $v_2$. For convenience, we use the second equation, and find that a solution must satisfy

$$v_1 = (1 - j)v_2,$$

thus all eigenvectors are obtained as

$$v_1 = (1 - j)c, \quad v_2 = c,$$

i.e.,

$$\mathbf{v}^{(1)} = \begin{pmatrix} (1 - j)c \\ c \end{pmatrix} = c \begin{pmatrix} 1 - j \\ 1 \end{pmatrix},$$
where \( c \) is an arbitrary complex constant. We can take any specific nonzero value for \( c \), and for convenience we take \( c = 1 \), then

\[
\mathbf{v}^{(1)} = \begin{pmatrix} 1 - j \\ 1 \end{pmatrix}.
\]

(Alternative choices for \( \mathbf{v}^{(1)} \) include

\[
\mathbf{v}^{(1)} = \begin{pmatrix} 1/2 + j/2 \\ 0 \end{pmatrix}, \quad \text{or} \quad \mathbf{v}^{(1)} = \begin{pmatrix} 0 \\ 1 + j \end{pmatrix},
\]

and there are many others, all of which will give a correct general solution in the end.)

iii) A complex-valued solution is

\[
e^{rt} \mathbf{v}^{(1)} = e^{(-2+j)t} \begin{pmatrix} 1 - j \\ 1 \end{pmatrix} = e^{-2t} e^{jt} \begin{pmatrix} 1 - j \\ 1 \end{pmatrix} = e^{-2t} (\cos t + j \sin t) \begin{pmatrix} 1 - j \\ 1 \end{pmatrix}
\]

which has real and imaginary parts

\[
\mathbf{u}(t) = e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix}, \quad \mathbf{v}(t) = e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}.
\]

iv) Therefore the general solution is

\[
\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -\cos t + \sin t \\ \sin t \end{pmatrix}
\]

or

\[
x_1(t) = (c_1 - c_2) e^{-2t} \cos t + (c_1 + c_2) e^{-2t} \sin t, \quad x_2(t) = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t,
\]

where \( c_1 \) and \( c_2 \) are arbitrary (real) constants. (There are many other possible forms for a general solution, depending on the choice of eigenvector \( \mathbf{v}^{(1)} \), for example

\[
\mathbf{x}(t) = c_1 e^{-2t} \begin{pmatrix} 2 \cos t \\ \cos t - \sin t \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \sin t \\ \cos t + \sin t \end{pmatrix}
\]

is just one alternate form for a general solution.)

(c) \( x' = -\frac{3}{2} x + y, \quad y' = -\frac{1}{4} - \frac{1}{2} y. \)

i) Write the system in vector form

\[
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.
\]
The characteristic equation is

\[ 0 = \begin{vmatrix} -\frac{3}{2} - r & 1 \\ -\frac{1}{4} & -\frac{1}{2} - r \end{vmatrix} = r^2 + 2r + 1 = (r + 1)^2, \]

so the eigenvalues are repeated

\[ r_1 = -1, \quad r_2 = -1. \]

\( ii \) For \( r_1 = r_2 = -1 \), a corresponding eigenvector \( \mathbf{v} = (v_1 \quad v_2)^T \) must satisfy

\[
\begin{pmatrix}
-\frac{3}{2} - (-1) & 1 \\
-\frac{1}{4} & -\frac{1}{2} - (-1)
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= 
\begin{pmatrix}
-\frac{1}{2} & 1 \\
-\frac{1}{4} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

or

\[
-\frac{1}{2} v_1 + v_2 = 0, \\
-\frac{1}{4} v_1 + \frac{1}{4} v_2 = 0,
\]

Noting that one equation is just a constant multiple of the other equation, we just have to satisfy one of them, so all solutions are given by

\[ v_1 = c, \quad v_2 = \frac{1}{2} c \]

with \( c \) arbitrary, i.e.,

\[ \mathbf{v} = c \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} . \]

We see there is only one linearly independent eigenvector corresponding to the eigenvalue \( r_1 = r_2 = 1 \). For convenience, we take \( c = 2 \), then

\[ \mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} . \]

\( iii \) Then we find a generalized eigenvector \( \mathbf{w} = (w_1 \quad w_2)^T \), satisfying

\[
\begin{pmatrix}
-\frac{1}{2} & 1 \\
-\frac{1}{4} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
1
\end{pmatrix},
\]

or equivalently

\[
-\frac{1}{2} w_1 + w_2 = 2, \\
-\frac{1}{4} w_1 + \frac{1}{2} w_2 = 1,
\]

and again one equation is a constant multiple of the other equation. Thus we only need to satisfy one of the equations, say \(-(1/2)w_1 + w_2 = 2\) and all possible solutions can be characterized as

\[ w_1 = c, \quad w_2 = 2 + \frac{1}{2} c \]
or
\[
\mathbf{w} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1/2 \end{pmatrix},
\]
where \( c \) is an arbitrary constant. We can take, for example, \( c = 0 \) here and then a specific generalized eigenvector is
\[
\mathbf{w} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.
\]

iv) The general solution is
\[
\mathbf{x}(t) = c_1 e^{-t} \mathbf{v} + c_2 e^{-t} [\mathbf{w} + \mathbf{v} t],
\]
or
\[
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \left[ \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} t \right]
\]

in vector form, or
\[
x(t) = 2 c_1 e^{-t} + 2 c_2 t e^{-t}, \quad y(t) = c_1 e^{-t} + c_2 (2 + t) e^{-t}
\]
in component form, where \( c_1 \) and \( c_2 \) are arbitrary constants. (There are many other possible forms for a general solution.)

3. The system \( i' = \frac{1}{L} v, \quad v' = -\frac{1}{c} i - \frac{1}{RC} v \) is written in matrix notation as
\[
\begin{pmatrix} i' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix}.
\]

(a) \( L = 1 \) henry, \( R = 1 \) ohm, \( C = 0.5 \) farad, \( i(0) = 2 \) amperes and \( v(0) = 1 \) volt.
i) Write the system as
\[
\begin{pmatrix} i' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix}.
\]
and solve the characteristic equation
\[
0 = \begin{vmatrix} -r & 1 \\ -2 & -2 - r \end{vmatrix} = r^2 + 2r + 2
\]
to find the eigenvalues \( r_{1,2} = (-2 \pm \sqrt{4 - 8})/2 \) are complex:
\[
r_1 = -1 + j, \quad r_2 = -1 - j.
\]

ii) For the complex eigenvalue \( r_1 = -1 + j \) we find a complex eigenvector \( \mathbf{v}^{(1)} = (v_1 \quad v_2)^T \), where
\[
\begin{pmatrix} -(1 + j) \\ -2 \end{pmatrix} \mathbf{v}^{(1)} = \begin{pmatrix} 1 - j \\ -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
or as an equivalent system of equations

\[
\begin{align*}
(1 - j)v_1 + v_2 &= 0, \\
-2v_1 - (1 + j)v_2 &= 0.
\end{align*}
\]

The second equation can be seen as \(-(1 + j)\) times the first equation, so we can use either equation to find a relation between \(v_1\) and \(v_2\). For convenience, we use the first equation, and find that a solution must satisfy

\[
v_2 = (-1 + j)v_2,
\]

thus all eigenvectors are obtained as

\[
v_1 = c, \quad v_2 = (-1 + j)c,
\]

i.e.,

\[
v^{(1)} = \begin{pmatrix} c \\ (-1 + j)c \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 + j \end{pmatrix},
\]

where \(c\) is an arbitrary complex constant. We can take any specific nonzero value for \(c\), and for convenience we take \(c = 1\), then

\[
v^{(1)} = \begin{pmatrix} 1 \\ -1 + j \end{pmatrix}.
\]

(Alternative choices for \(v^{(1)}\) include

\[
v^{(1)} = \begin{pmatrix} -1 + j/2 \\ 1 \end{pmatrix}, \quad \text{or} \quad v^{(1)} = \begin{pmatrix} -1 - j/2 \\ 2 \end{pmatrix},
\]

and there are infinitely many others, all of which will give a correct general solution.)

\(iii\) A complex-valued solution is (use \(j^2 = -1\))

\[
e^{rt}v^{(1)} = e^{(-1+j)t} \begin{pmatrix} 1 \\ -1 + j \end{pmatrix} = e^{-t}e^{jt} \begin{pmatrix} 1 \\ -1 + j \end{pmatrix} \\
= e^{-t}(\cos t + j \sin t) \begin{pmatrix} 1 \\ -1 + j \end{pmatrix} \\
= e^{-t} \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + je^{-t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}.
\]

\(iv\) Therefore the general solution is

\[
\begin{pmatrix} i(t) \\ v(t) \end{pmatrix} = c_1e^{-t} \begin{pmatrix} \cos t \\ -\cos t - \sin t \end{pmatrix} + c_2e^{-t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}
\]

or

\[
i(t) = c_1e^{-t} \cos t + c_2e^{-t} \sin t, \quad v(t) = (-c_1 + c_2)e^{-t} \cos t + (-c_1 - c_2)e^{-t} \sin t,
\]
where \( c_1 \) and \( c_2 \) are arbitrary (real) constants. (There are many other possible forms for a general solution, depending on the choice of eigenvector \( \mathbf{v}^{(1)} \).

v) Evaluating the general solution at \( t = 0 \) and using the initial conditions, we have

\[
c_1 = 2, \quad -c_1 + c_2 = 1
\]

which is easily solved,

\[
c_1 = 2, \quad c_2 = 3.
\]

Therefore the current through the inductor and the voltage across the capacitor are

\[
i(t) = e^{-t}(2\cos t + 3\sin t) \text{ amperes,} \quad v(t) = e^{-t}(\cos t - 5\sin t) \text{ volts.}
\]

(Although the general solution can take many forms, the solution to the initial value problem is unique.)

(b) \( L = 4 \) henries, \( R = 1 \) ohm, \( C = 1 \) farad, \( i(0) = 1 \) ampere and \( v(0) = 2 \) volts.

i) Write the system as

\[
\begin{pmatrix}
i' \\
v'
\end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ -1 & -1 \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix}.
\]

The characteristic equation is

\[
0 = \begin{vmatrix} -r & \frac{1}{4} \\ -1 & -1 - r \end{vmatrix} = r^2 + r + \frac{1}{4} = (r + \frac{1}{2})^2,
\]

so the eigenvalues are repeated

\[
r_1 = -\frac{1}{2}, \quad r_2 = -\frac{1}{2}.
\]

ii) For \( r_1 = r_2 = -1/2 \), a corresponding eigenvector \( \mathbf{v} = (v_1 \ v_2)^T \) must satisfy

\[
\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -1 + \frac{1}{2} \end{pmatrix} \mathbf{v} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ -1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

or

\[
\frac{1}{2} v_1 + \frac{1}{4} v_2 = 0, \\
-v_1 - \frac{1}{2} v_2 = 0,
\]

Noting that one equation is just a constant multiple of the other equation, we just have to satisfy one of them, so all solutions are given by

\[
v_1 = c, \quad v_2 = -2c
\]

with \( c \) arbitrary, i.e.,

\[
\mathbf{v} = c \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]
We see there is only one linearly independent eigenvector corresponding to the eigenvalue \( r_1 = r_2 = -1/2 \). For convenience, we take \( c = 1 \), then

\[
v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]

(iii) Then we find a generalized eigenvector \( w = (w_1 \ w_2)^T \), satisfying

\[
\begin{pmatrix}
\frac{1}{2} & \frac{3}{4}
\end{pmatrix}
\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},
\]

or equivalently

\[
\begin{align*}
\frac{1}{2} w_1 + \frac{3}{4} w_2 &= 1, \\
-w_1 - \frac{1}{2} w_2 &= -2,
\end{align*}
\]

and again one equation is a constant multiple of the other equation. Thus we only need to satisfy one of the equations, say \(-w_1 - (1/2)w_2 = -2\), or \(w_1 + (1/2)w_2 = 2\), and all possible solutions can be characterized as

\[
w_1 = c, \quad w_2 = 4 - 2c
\]

or

\[
w = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + c \begin{pmatrix} 1 \\ -2 \end{pmatrix},
\]

where \( c \) is an arbitrary constant. We can take, for example, \( c = 0 \) here and then a specific generalized eigenvector is

\[
w = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.
\]

(iv) The general solution is

\[
\begin{pmatrix} i(t) \\ v(t) \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t/2} \left[ \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} t \right]
\]

in vector form, or

\[
i(t) = c_1 e^{-t/2} + c_2 t e^{-t/2}, \quad v(t) = -2c_1 e^{-t/2} + c_2 (4 - 2t) e^{-t/2}
\]

in component form, where \( c_1 \) and \( c_2 \) are arbitrary constants.

(v) Evaluating the general solution at \( t = 0 \) and using the initial conditions, we have

\[
c_1 = 1, \quad -2c_1 + 4c_2 = 2
\]

from which

\[
c_1 = 1, \quad c_2 = 1.
\]

Therefore the current through the inductor and the voltage across the capacitor are

\[
i(t) = (1 + t) e^{-t/2} \text{ amperes}, \quad v(t) = 2(1 - t) e^{-t/2} \text{ volts}.
\]