1. For each function below, say whether the function is even, odd or neither.

   (a) \( \sin(5x) \)
   (b) \( \sin(5x) + \cos(5x) \)
   (c) \( x^3 \cos(17x) \)
   (d) \( x^{2/3} \)
   (e) \( e^x - e^{-x} \)

   a. odd: \( \sin(-5x) = -\sin(5x) \).
   b. neither: Let \( f(x) := \sin(5x) + \cos(5x) \). Then \( f(-x) = \sin(-5x) + \cos(-5x) = -\sin(5x) + \cos(5x) \). So, \( f(0) = 1 \) and \( f(-0) = f(0) = 1 \neq -1 \), and so it is not odd. And \( f(\pi/2) = -1 \), while \( f(-\pi/2) = 1 \), and so is not even.
   c. odd, since it is the product of an odd function and an even function.
   d. even since \((-x)^{2/3} = x^{2/3}\)
   e. odd since \( e^x - e^{-x} = -(e^x - e^{-x}) \)

2. Find Fourier series for each of the following functions.

   (a) \( f(x) = \sin(\pi x) + \cos(3\pi x) \)
   (b) \( f(x) = \sin(3x) \)

   a. \( \sin(\pi x) \) has fundamental period 2 and \( \cos(3\pi x) \) has fundamental period 2/3. Since 2 is an integer multiple of 2/3, \( f(x) \) has fundamental period 2. So, \( L = 1 \) and the Fourier series is of the form

   \[
   \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\pi x) + b_m \sin(m\pi x)
   \]

   But \( f(x) = \sin(\pi x) + \cos(3\pi x) \) is of this form and so it is a Fourier series of itself (one can show, using the orthogonality relations, that the Fourier series of a function is unique)

   b. \( \sin(3x) \) has fundamental period 2\pi/3 and so \( L = \pi/3 \) and the Fourier series is of the form

   \[
   \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(3mx) + b_m \sin(3mx)
   \]

   But \( f(x) = \sin(3x) \) is of this form and it is a Fourier series of itself.

3. Find the Fourier series for the following function and sketch the graph of the function to which the series converges over three fundamental periods.

   \[ f(x) = \begin{cases} 
   0 & -2 \leq x \leq 0 \\
   2 - x & 0 < x \leq 2 
   \end{cases} \]
\[ f(x + 4) = f(x). \]

Since the fundamental period is 4, \( L = 2 \). By the Euler-Fourier formulas,

\[ a_n = (1/2) \int_{-2}^{2} f(x) \cos(n \pi x/2) = (1/2) \int_{0}^{2} (2 - x) \cos(n \pi x/2) \]

Now,

\[ \int_{0}^{2} \cos(n \pi x/2) = \frac{2}{n \pi} \sin(n \pi x/2)|_0^2 = 0. \]

By integration by parts, for \( n \geq 1 \),

\[ \int_{0}^{2} x \cos(n \pi x/2) = \frac{4}{n^2 \pi^2} \left( \cos(n \pi x/2) + (n \pi/2)x \sin(n \pi x/2) \right)|_0^2 \]

\[ = \left( \frac{4}{n^2 \pi^2} \right)((-1)^n - 1) \]

Combining the previous two statements, we get for \( n \geq 1 \),

\[ a_n = \left( \frac{2}{n^2 \pi^2} \right)(1 - (-1)^n) \]

And

\[ a_0 = (1/2) \int_{0}^{2} (2 - x) = (1/2)(2x - (1/2)x^2)|_0^2 = 1 \]

And

\[ b_n = (1/2) \int_{-2}^{2} f(x) \sin(n \pi x/2) = (1/2) \int_{0}^{2} (2 - x) \sin(n \pi x/2) \]

Now,

\[ \int_{0}^{2} \sin(n \pi x/2) = -\frac{2}{n \pi} \cos(n \pi x/2)|_0^2 = -\frac{2}{n \pi}((-1)^n - 1) = \frac{2}{n \pi}(1 - (-1)^n) \]

By integration by parts

\[ \int_{0}^{2} x \sin(n \pi x/2) = \frac{4}{n^2 \pi^2} \left( \sin(n \pi x/2) - (n \pi/2)x \cos(n \pi x/2) \right)|_0^2 \]

\[ = -\frac{4}{n^2 \pi^2}(n \pi/2)(-1)^n = -\frac{4}{n \pi}(-1)^n \]

So the Fourier series is

\[ f(x) = 1/2 + \sum_{n=1}^{\infty} \left( \frac{2}{n^2 \pi^2} \right)(1 - (-1)^n) \cos(n \pi x/2) + \sum_{n=1}^{\infty} \frac{2}{n \pi} \sin(n \pi x/2) \]

4. Let \( f(x) = x - 1, \ 0 < x \leq 1. \)
(a) Define extensions of \( f \) to both an even function and an odd function of period 2 on \( \mathbb{R} \), and sketch both extensions.

(b) Find the Fourier series for each extension.

a. Odd extension: 
\[ f(x) = \begin{cases} 
  x - 1 & 0 < x \leq 1 \\
  0 & x = 0 \\
  x + 1 & -1 \leq x < 0 
\end{cases} \]

Even extension: 
\[ f(x) = \begin{cases} 
  x - 1 & 0 < x \leq 1 \\
  -x - 1 & -1 \leq x < 0 
\end{cases} \]

b. For odd extension, we get a Fourier sine series
\[ f(x) = \sum_{n=1}^{\infty} b_n \sin(n \pi x) \]

where
\[ b_n = 2 \int_{0}^{1} (x - 1) \sin(n \pi x) = 2 \int_{0}^{1} x \sin(n \pi x) - 2 \int_{0}^{1} \sin(n \pi x) \]
\[ = \frac{2}{n \pi} (\sin(n \pi x) - (n \pi x)(\cos(n \pi x)))\big|_{0}^{1} - \frac{2}{n \pi} \sin(n \pi x)\big|_{0}^{1} \]
\[ = -\frac{2}{n \pi} (-1)^n + \frac{2}{n \pi}((-1)^n - 1) = -\frac{2}{n \pi} \]

So,
\[ f(x) = \sum_{n=1}^{\infty} \frac{-2}{n \pi} \sin(n \pi x) \]

For even extension, we get a Fourier cosine series
\[ f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n \pi x) \]

where, for \( n \geq 1, \)
\[ a_n = 2 \int_{0}^{1} (x - 1) \cos(n \pi x) = 2 \int_{0}^{1} x \cos(n \pi x) - 2 \int_{0}^{1} \cos(n \pi x) \]
\[ = \frac{2}{n^2 \pi^2} (\cos(n \pi x) + (n \pi x)(\sin(n \pi x)))\big|_{0}^{1} - \frac{2}{n \pi} \sin(n \pi x)\big|_{0}^{1} \]
\[ = \frac{2}{n^2 \pi^2}((-1)^n - 1) \]

and
\[ a_0 = 2 \int_{0}^{1} (x - 1) dx = x^2 - 2x\big|_{0}^{1} = -1 \]
So,

\[ f(x) = -1/2 + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} ((-1)^n - 1) \cos(n \pi x) \]
Sketch for #3
Sketch for #4a

Odd Extension:

Even Extension: