6.2.3 continued

Example 6.2.E continued

1. \[ x'' + 4x = t - (t-\pi)u(t-\pi), \quad x(0) = 0, \quad x'(0) = 0 \]

2. Next, take Laplace transforms, and algebraically solve for the transform of the solution.

Let \[ \mathcal{L}\{x(t)\} = X(s) \]

\[ \mathcal{L}\{x''(t)\} + 4\mathcal{L}\{x(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{(t-\pi)u(t-\pi)\} \]

\[
\begin{align*}
    s^2X(s) - s x(0) - x'(0) + 4 X(s) &= \frac{1}{s^2} - e^{-\pi s} \frac{1}{s^2} \\
    s^2 X(s) + 4 X(s) &= \frac{1}{s^2} - e^{-\pi s} \frac{1}{s^2} \\
    X(s) &= \frac{1}{s^2(s^2+4)} - e^{-\pi s} \frac{1}{s^2(s^2+4)}
\end{align*}
\]

3. Find the inverse transform \[ x(t) = \mathcal{L}^{-1}\{X(s)\} \]

- partial fractions: \[ \frac{1}{s^2(s^2+4)} \]

- second shifting property: \[ e^{-\pi s} F(s) \]

Partial fractions

\[
\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}
\]

from general linear factor

from simple (irreducible) quadratic factor
\[
1 = A s (s^2 + 4) + B (s^2 + 4) + (Cs + D) s^2 \\
1 = (A + C) s^3 + (B + D) s^2 + 4As + 4B \\
\Rightarrow \quad B = \frac{1}{4}, \quad A = 0, \quad D = -\frac{1}{4}, \quad C = 0
\]

\[
\frac{1}{s^2(s^2 + 4)} = \frac{1}{4} \frac{1}{s^2} - \frac{1}{4} \frac{1}{s^2 + 4}
\]

\[
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 4)}\right\} = \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\}
\]

\[
= \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{4} \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2 + 2^2}\right\}
\]

\[
= \frac{1}{4} t - \frac{1}{8} \sin(2t)
\]

Using this, and the second shifting property:

\[
\mathcal{L}^{-1}\left\{e^{-\tau s} \frac{1}{s^2(s^2 + 4)}\right\} = \left[\frac{1}{4} (t-\pi) - \frac{1}{8} \sin(2(t-\pi))\right] u(t-\pi)
\]

Solution of IVP (written in terms of Heaviside function):

\[
x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 4)}\right\} - \mathcal{L}^{-1}\left\{e^{-\tau s} \frac{1}{s^2(s^2 + 4)}\right\}
\]

\[
x(t) = \frac{1}{4} t - \frac{1}{8} \sin(2t)
\]

\[
-\left[\frac{1}{4} (t-\pi) - \frac{1}{8} \sin(2(t-\pi))\right] u(t-\pi)
\]

= 0 or 1
(written as a piecewise defined function)

\[
x(t) = \begin{cases} 
\frac{1}{4}t - \frac{1}{8}\sin(2t) & \text{if } 0 \leq t < \pi \\
\frac{1}{4}t - \frac{1}{8}\sin(2t) - \frac{1}{4}(t-\pi) + \frac{1}{8}\sin(2(t-\pi)) & \text{if } t \geq \pi 
\end{cases}
\]

(and simplified) \[ \sin(2(t-\pi)) = \sin(2t - 2\pi) = \sin(2t) \]

\[
x(t) = \begin{cases} 
\frac{1}{4}t - \frac{1}{8}\sin(2t) & \text{if } 0 \leq t < \pi \\
\frac{\pi}{4} & \text{if } t \geq \pi 
\end{cases}
\]

**Exercise**  Show that \( x''(t) \) is continuous, \( x'''(t) \) is not

(Compare: \( f(t) \) is continuous, \( f'(t) \) is not)

**Exercise**  In Ex. 6.2.C, show that \( x'(t) \) is continuous, \( x''(t) \) is not (Compare: \( f(t) \) is not continuous)

These two Exercises may give you some intuition into what properties to expect for a solution \( x(t) \), of \( Lx = f(t) \), when \( f(t) \) is piecewise defined.

Graph of solution for \( t \geq 0 \): next page

Note that \( \sin(2t) \) has a period of \( \pi \):

\[
\begin{array}{c}
\text{time-consuming}
\end{array}
\]

and recall \( x(0) = 0, \ x'(0) = 0 \)

This will help to draw the graph, without a calculator or Calc I methods.
Graph of solution $x(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t) - \left[\frac{1}{4}(t - \pi) - \frac{1}{8}\sin(2(t - \pi))\right] u(t - \pi)$

Note: Scales on horizontal and vertical axes are different.