\( \theta' = \omega, \quad \omega' = -\frac{g}{L} \sin(\theta) \) (continued)

5. Draw the phase portrait:

Because \( \theta'' + \frac{g}{L} \sin(\theta) = 0 \) is conservative, there are extra steps.

Trajectories \((\theta(t), \omega(t))\) of the corresponding system must lie on curves given implicitly by

\[
\frac{1}{2} \omega^2 + \int \frac{g}{L} \sin(\theta) \, d\theta = C
\]

\[
H(\theta, \omega) = \frac{1}{2} \omega^2 - \frac{g}{L} \cos(\theta) = C
\]

The 3-D graph \( z = H(\theta, \omega) \) has parabolic cross-sections \( z = \frac{1}{2} \omega^2 + \text{constant} \) with min. at \( \omega = 0 \), for any fixed \( \theta = \Theta_0 \), we can find the contours \( H(\theta, \omega) = C \) from a plot of \( z = H(\theta, 0) \):
Contours of $H(\theta, \omega)$ in $(\theta, \omega)$-plane

Crit. pts. $(n\pi, 0)$, $n$ even integer $\circ$ are local minima of $z = H(\theta, \omega)$

Crit. pts. $(n\pi, 0)$, $n$ odd integer $\bullet$ are saddle points
Phase portrait of $\Theta' = \omega$, $\omega' = -\frac{g}{l}\sin(\Theta)$ is (axes, nullclines not shown, see 27 p. 8)

Picture repeats in $\Theta$-direction with period $2\pi$.

"Low energy" trajectories near stable crit. pt. (0,0) are periodic, but period depends on amplitude (unlike the linear centre for $\Theta' = \omega$, $\omega' = -\frac{g}{l}\Theta$, where the period is constant)

See textbook pp. 366-367 esp. Fig. 8.8, but you don't need to know details of derivation. These "low energy" trajectories ($H(\Theta, \omega) < \frac{g}{l}$) represent back-and-forth oscillations of $\Theta(t)$ about $\Theta = 0$. What about if $H(\Theta, \omega) > \frac{g}{l}$?