Example 8.2.3: Draw the phase portrait for the system of 1st order equations that is equivalent to

\[ x'' + x - x^2 = 0 \]

Recognize as conservative equation.

Write as system

\[
\begin{cases}
  x' = y \\
  y' = -x + x^2
\end{cases}
\]

Exercise Find all critical points: \((0,0), (1,0)\)

Linearize at each critical point and determine

\((1,0)\) is a saddle, unstable "linear centre"

\((0,0)\) has purely imaginary eigenvalues, linearization fails to determine behaviour or stability

Draw the nullclines and direction field, draw the local phase portrait near \((1,0)\) only.
Now using the fact that this system comes from a conservative equation
\[ x'' + \frac{x - x^2}{f(x)} = 0 \]
we know the trajectories of the system \((x(t), y(t))\) lie on curves given implicitly by
\[ \frac{1}{2} y^2 + \int (x - x^2) \, dx = C \]
\[ \frac{1}{2} y^2 + \frac{1}{2} x^2 - \frac{1}{3} x^3 = C \]

\[ H(x, y) \] "Hamiltonian" or "energy" function

Need to draw \( H(x, y) = C \) in \((x, y)\) plane.

Set \( y = 0 \), draw 2-D graph of \( z = H(x, 0) \) in \((x, z)\)-plane: "potential energy"

\[ z = \frac{1}{2} x^2 - \frac{1}{3} x^3 \]
\[ \left( \frac{dz}{dx} = x - x^2 \right) \]
Optional: set $x = x_0$ (constant), draw 2-D graph of $z = H(x_0, y)$ in $(y, z)$-plane

$z = \frac{1}{2} y^2 + \text{constant}$

\[ \frac{1}{2} x_0^2 - \frac{1}{3} x_0^3 \]

min. has value $\frac{1}{2} x_0^2 - \frac{1}{3} x_0^3$

Optional: draw 3-D graph of $z = H(x, y)$ in $(x, y, z)$-space

paraboloid cross-sections

local min above $(x, y) = (0, 0)$

saddle point above $(x, y) = (1, 0)$
Short cut to $H(x,y) = C$ graphs ("contours")

"side view"
$z = c_1, c_1 > \frac{1}{6}$
$z = \frac{1}{6}$
$z = c_0, 0 < c_0 < \frac{1}{6}$

"top view"
$H(x,y) = c_1$
$0 < c_1 < \frac{1}{6}$
$H(x,y) = \frac{1}{6}$

Contours of $H(x,y)$
Phase portrait of nonlinear system: must be consistent with nullclines, direction field, local phase portraits and graphs of contours $H(x,y) = C$

Now we see that $(0,0)$ is (Lyapunov) stable but not asymptotically stable: it is called a "nonlinear centre"

- corresponds to local min. of potential energy for conservative eqn. $z = \frac{1}{2}x^2 - \frac{1}{3}x^3$