8.1.2 Linearization

If \((x_0, y_0)\) is a critical point, define \(u(t), v(t)\) by

\[
x(t) = x_0 + u(t) \quad (u = x - x_0)
\]
\[
y(t) = y_0 + v(t) \quad (v = y - y_0)
\]

where \((x(t), y(t))\) is a solution of (8.1).

Find new ODEs for \(u, v\) assuming they are near 0

i) \(u' = x' = f(x, y) = f(x_0 + u, y_0 + v)\)

\[
= f(x_0, y_0) + \frac{df}{dx}(x_0, y_0) u
\]

\[
= 0 \quad \text{(why?)} + \frac{df}{dy}(x_0, y_0) v + \ldots
\]

\[
u' = \frac{df}{dx}(x_0, y_0) u + \frac{df}{dy}(x_0, y_0) v + \ldots
\]

remainder
of Taylor
series
\[ u' = \frac{df}{dx}(x_0, y_0) u + \frac{df}{dy}(x_0, y_0) v + \ldots \]

\[
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{df}{dx}(x_0, y_0) & \frac{df}{dy}(x_0, y_0) \\ \frac{dg}{dx}(x_0, y_0) & \frac{dg}{dy}(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \ldots
\]

**Jacobian matrix** (or derivative) of \[ \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \], evaluated at \((x_0, y_0)\).

By ignoring all the remainder terms we get **linear approximation** of (8.1), for \((x, y)\) near \((x_0, y_0)\), called the **linearization** of (8.1) at \((x_0, y_0)\):

\[
\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{df}{dx}(x_0, y_0) & \frac{df}{dy}(x_0, y_0) \\ \frac{dg}{dx}(x_0, y_0) & \frac{dg}{dy}(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]
Example 8.1. A, cont.

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} y \\ -y - x + x^2 \end{bmatrix}
\]

(2) Find the linearization at each critical point.

Compute the Jacobian matrix:

\[
\begin{bmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x & -1 \end{bmatrix}
\]

(a) At the critical point \((x_0, y_0) = (0, 0)\)

\((-1 + 2x_0 = -1 + 2(0) = -1)\)

the linearization is

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]

(b) At the critical point \((x_0, y_0) = (1, 0)\)

\((-1 + 2(1) = 1)\)

\[
\begin{bmatrix}
u' \\
v'
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\]
8.2 Stability and classification of isolated critical points

8.2.1 Isolated critical points and almost linear systems

A critical point \((x_0, y_0)\) is isolated if it is the only critical point in some sufficiently small open rectangle that contains \((x_0, y_0)\).

e.g. \[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
y \\
0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
y \\
x
\end{bmatrix}
\]

has a critical point \((x_0, y_0) = (0, 0)\)

But, all crit. pts. are solns. of \[\begin{cases}
y = 0 \\
x = 0
\end{cases}\]

The entire x-axis consists of crit. pts.

\((0, 0)\) is not an isolated critical point
A system (8.1) is almost linear at a critical point \((x_0,y_0)\) if \((x_0,y_0)\) is isolated and the Jacobian matrix evaluated at \((x_0,y_0)\) is invertible (i.e. nonsingular i.e does not have a 0 eigenvalue) e.g. p. 357

8.2.2 Stability and classification of isolated critical points

Let \((x_0,y_0)\) be a critical point of (8.1).

(a) \((x_0,y_0)\) is (Lyapunov) stable if every solution \((x(t),y(t))\), that starts at \(t \geq 0\) with initial value \((x(0),y(0))\) sufficiently close to \((x_0,y_0)\), remains arbitrarily close to \((x_0,y_0)\) for all \(t \geq 0\).
(b) \((x_0, y_0)\) is unstable if it is not \((\text{Lyapunov})\) stable \((\text{i.e. at least one soln. starts at } t=0 \text{ with initial value arbitrarily close to } (x_0, y_0) \text{ but does not remain sufficiently close for all } t \geq 0)\).

(c) \((x_0, y_0)\) is asymptotically stable if every solution, that starts at \(t=0\) with initial value sufficiently close to \((x_0, y_0)\), approaches \((x_0, y_0)\) as \(t \to \infty\):

\[
\lim_{t \to \infty} x(t) = x_0 \\
\lim_{t \to \infty} y(t) = y_0
\]

In "most" cases, for (8.1) the local behavior and stability at crit. pts. can be determined by the eigenvalues of the linearization.
<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>(Local) Behaviour</th>
<th>Stability</th>
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<tbody>
<tr>
<td>real distinct positive</td>
<td>(nodal) source/ unstable node</td>
<td>unstable</td>
</tr>
<tr>
<td>real distinct negative</td>
<td>(nodal) sink/ stable node</td>
<td>asymptotically stable</td>
</tr>
</tbody>
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