Typical graph

\[ x(t) = x(t) = x_{np}(t) + x_{sp}(t) \]

\[ x(t) \approx x_{sp}(t) \]

\[ x = x_{sp}(t) = C \cos(\omega t - \phi) \]

If \( m, c, k, F_0 \) are fixed, and consider \( \omega > 0 \) as a parameter, the \( A, B \) will be expressions involving \( \omega \) : \( A = A(\omega), \ B = B(\omega) \)

\[ C = C(\omega) = \sqrt{A(\omega)^2 + B(\omega)^2} \]

If \( c > 0 \) is small,

As \( c \to 0 \), \( C_{max} \to \infty \)

near (but not =) \( \omega_0 = \sqrt{\frac{k}{m}} \)

L "practical resonance frequency"
Chapter 3  Systems of ODEs

3.1 Introduction to systems of ODEs

First order systems

\[
\begin{cases}
  x_1' = g_1(x_1, x_2, \ldots, x_n, t) \\
  x_2' = g_2(x_1, x_2, \ldots, x_n, t) \\
  \vdots \\
  x_n' = g_n(x_1, x_2, \ldots, x_n, t)
\end{cases}
\]

In vector notation

\[
\vec{x}' = \vec{g}(\vec{x}, t), \quad \vec{x} \in \mathbb{R}^n
\]

A solution \( \vec{x}(t) = (x_1(t) \ldots x_n(t))^T \) is continuous, and satisfies the system of ODEs when it is substituted in.

Example 3.1. Write \( y'' + p(t)y' + q(t)y = f(t) \) as an (equivalent) 1st order system.
Let \( x_1(t) = y(t) \), \( x_2(t) = y'(t) \). Then

\[
\begin{align*}
    x_1' &= y' = x_2 \\
    x_2' &= y'' = -q(t)y - p(t)y' + f(t) \\
    &= -q(t)x_1 - p(t)x_2 + f(t)
\end{align*}
\]

1st order system is

\[
\begin{cases}
    x_1' = x_2 \\
    x_2' = -q(t)x_1 - p(t)x_2 + f(t)
\end{cases}
\]

or

\[
\begin{bmatrix}
    x_1' \\
    x_2'
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    -q(t) & -p(t)
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} + \begin{bmatrix}
    0 \\
    f(t)
\end{bmatrix}
\]

A system \( \vec{x}' = \vec{g}(\vec{x}) \)

is autonomous if \( \vec{g}(\vec{x}) \) does not depend explicitly on \( (\text{the indep. variable}) t \).
For autonomous systems, it is meaningful to plot a vector field or direction field: at each point \( \mathbf{x} \), the vector \( \mathbf{g}(\mathbf{x}) \) specifies the "phase velocity" \( \mathbf{x}' \) and the "direction" vector \( \frac{x'}{||x'||} \) that is tangent to the solution trajectory \( \mathbf{x}(t) \) in \( \mathbb{R}^n \).

**Example 3.1.B**

\[
\begin{align*}
\mathbf{x}' &= \mathbf{g}(\mathbf{x}) = \begin{bmatrix} -1 + 2(0) \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
\text{(direction is } \frac{x'}{||x'||} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix} \text{)}
\end{align*}
\]

is autonomous. At the point \( \mathbf{x} = (1, 0) \) the vector is \( \mathbf{x}' = \mathbf{g}(\mathbf{x}) \).
Do this for many points \( \vec{x} \) (e.g. in a grid) to get an overall picture of the vector field or direction field.

3.3 Linear systems of ODEs

\[
\begin{align*}
\dot{\vec{x}} &= P(t) \vec{x} + \vec{f}(t) \\
\text{e.g. } (n=2) & \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2t & e^t \\ -1 & 1/t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}
\end{align*}
\]

Homogeneous if \( \vec{f}(t) \equiv 0 \)
\[ \mathbf{x}' = P(t) \mathbf{x} \]

**Theorem 3.3.1** If \( P(t) \) is a continuous \( n \times n \) matrix and if (\(*)\) has \( n \) linearly independent solutions \( \mathbf{x}_1(t), ..., \mathbf{x}_n(t) \) then the general solution of (\(*\)) is

\[
\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + \ldots + c_n \mathbf{x}_n(t)
\]

where \( c_1, ..., c_n \) are arbitrary constants.

Vector functions \( \mathbf{x}_1(t), ..., \mathbf{x}_n(t) \) are linearly independent in an interval \( I \) if

\[
c_1 \mathbf{x}_1(t) + \ldots + c_n \mathbf{x}_n(t) = 0 \quad \text{for all} \ t \ \text{in} \ I
\]

has only the solution \( c_1 = 0, ..., c_n = 0 \) i.e. \( \equiv 0 \)

If \( n = 2 \) and \( \mathbf{x}_1(t), \mathbf{x}_2(t) \) are solutions of (\*) they are linearly indep. if and only if one is not a constant multiple of the other.