### 3. B Forced periodic vibrations

**Forced with damping** \((\gamma > 0)\)

\[
m u'' + \gamma u' + ku = F_0 \cos(\omega t)
\]

or \(F_0 \sin(\omega t)\)

**Forcing frequency**

**Forcing amplitude**

**Non homogeneous linear eqn.**

**Gen. soln. of corresp. homog. eqn.**

\[
u_c(t) = c_1 u_1(t) + c_2 u_2(t)
\]

(3 cases for complementary solution)

**Particular solution** (method of undet. coeff.)

\[
U(t) = A \cos(\omega t) + B \sin(\omega t)
\]

(why "\(s = 0\)"?)

**Gen. soln. of nonhomogeneous eqn.**

\[
u(t) = u_c(t) + U(t)
\]

\[
= c_1 u_1(t) + c_2 u_2(t) + A \cos(\omega t) + B \sin(\omega t)
\]

\(\rightarrow 0 \text{ as } t \rightarrow \infty\)

\(R \cos(\omega t - \phi)\)

forced response

or "steady-state" soln.

**Transient solution**
\[ u(t) \approx U(t) \text{ for large } t \]

A, B can be determined in terms of \( m, \gamma, k, F_0, \omega \) (Exercise)

Then \( R = \sqrt{A^2 + B^2} = \text{amplitude of forced response} \)

\[ \tan \theta = \frac{B}{A} \]

After a lot of algebra

\[ R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}} \]

where \( \omega_0 = \sqrt{\frac{k}{m}} \)

If \( \gamma^2 < 2mk \), it can be shown that the maximum of \( R = R(\omega) \) for fixed \( F_0, m, k, \gamma \) is attained at

\[ \omega = \omega_{\text{max}} = \omega_0 \sqrt{1 - \frac{\gamma^2}{2mk}} \]  \hspace{1cm} \text{(Exercise)}

and

\[ R_{\text{max}} = R(\omega_{\text{max}}) = \frac{F_0}{8m_0 \sqrt{1 - \frac{\gamma^2}{4mk}}} \]
For fixed $m, k, F_0$: as $\gamma \to 0^+$ (small damping), $\omega_{\text{max}} \approx \omega_0$ natural freq. of mass+spring

$R_{\text{max}} \approx \frac{F_0}{\gamma \omega_0} \to \infty$

as $\gamma$ decreases

This is called resonance
Forced vibration without damping (ζ = 0)

\[ m u'' + k u = F_0 \cos(\omega t) \]

General solution of corresponding homogeneous eqn.

\[ u_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad \omega_0 = \sqrt{\frac{k}{m}} \]

Particular solution

\[ U(t) = A \cos(\omega t) + B \sin(\omega t) \quad \omega \neq \omega_0 \]

\[ U(t) = t \left[ A \cos(\omega_0 t) + B \sin(\omega_0 t) \right] \quad \omega = \omega_0 \]

\[ = R t \cos(\omega_0 t - \delta) \]

If \( \omega \neq \omega_0 \), then \( A = \frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad B = 0 \) \( \text{(Exercise)} \)

\[ u(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \]

If \( u(0) = 0, \ u'(0) = 0 \) then

\[ c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0 \] \( \text{(Exercise)} \)

and motion is given by
\[ u(t) = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t) \]

\[ = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[ \cos(\omega t) - \cos(\omega_0 t) \right] \]

\[ = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[ -2 \sin\left(\frac{\omega t + \omega_0 t}{2}\right) \sin\left(\frac{\omega t - \omega_0 t}{2}\right) \right] \]

\[ = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right) \]

If \( \omega \neq \omega_0 \) then \( \left| \frac{2F_0}{m(\omega_0^2 - \omega^2)} \right| \) is large

\( \left| \frac{\omega_0 - \omega}{2} \right| \) is small (slow frequency)

\( \left| \frac{\omega_0 + \omega}{2} \right| \approx \omega_0 \) is relatively large (fast frequency)

\[ u(t) = \left[ \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \right] \sin\left(\frac{\omega_0 + \omega}{2} t\right) \]

slowly varying

"amplitude"
\[
\begin{align*}
\text{If } w &= w_0 \quad \text{(Example).} \\
u(t) &= c_1 \cos(w_0 t) + c_2 \sin(w_0 t) + \left[ \frac{F_0}{2 mn_0} \right] \sin(\omega t) \\
\text{bounded} & \quad \text{unbounded}
\end{align*}
\]
\[ u = \frac{F_0}{2m\omega_0} \cdot t \]

\[ u = \frac{F_0}{2m\omega_0} \cdot t \sin(\omega t) \]

\[ u = -\frac{F_0}{2m\omega_0} \cdot t \]