1.8 Exact Equations

Suppose \( y(x) \) satisfies

\[
F(x, y(x)) = c \quad \text{ (implicit soln.)}
\]

Differentiate:

\[
\frac{d}{dx} F(x, y(x)) = \frac{\partial F}{\partial x}(x, y(x)) + \frac{\partial F}{\partial y}(x, y(x)) \frac{dy}{dx}(x) = 0
\]

ODE that \( y(x) \) satisfies

1.8.1 Solving exact equations

A 1st order ODE in the form

\[
M(x, y) + N(x, y) \frac{dy}{dx} = 0
\]

is exact if there exists \( F(x, y) \) such that

\[
M = \frac{\partial F}{\partial x} \quad \text{ and } \quad N = \frac{\partial F}{\partial y}
\]

In this case the solution is given
implicitly by
\[ F(x, y) = c \]

where \( c \) is an arbitrary constant.

How can we tell if an ODE is exact?

Idea: \( \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} \)

Theorem 1.8.1 If \( M(x, y) \) and \( N(x, y) \) are continuously differentiable and if
\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
\]
for all \((x, y)\) near a point \((x_0, y_0)\), then \( M + Ny' = 0 \) is exact near \((x_0, y_0)\).

Example 1.8. A
\[
\frac{dy}{dt} = - \frac{t^2 + y^2}{2(t+1)y} \quad , \quad y(0) = -1
\]

Write as
\[
\frac{t^2 + y^2}{M(t, y)} + \frac{2(t+1)y}{N(t, y)} \frac{dy}{dt} = 0
\]
Check if exact: is \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \)?

\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} (t^2 + y^2) = 2y \\
\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} 2(t+1)y = 2y
\]

equal, so ODE is exact.

Look for \( F(t, y) \) so that

\[
\frac{\partial F}{\partial t} = M = t^2 + y^2 \\
\text{and} \quad \frac{\partial F}{\partial y} = N = 2(t+1)y
\]

\[
\frac{\partial F}{\partial t} = t^2 + y^2 \Rightarrow \int (t^2 + y^2) \, dt \quad \text{(y constant)}
\]

\[
F = \frac{1}{3} t^3 + ty^2 + A(y) \quad \text{(1)}
\]

\[
\frac{\partial F}{\partial y} = 2ty + 2y \Rightarrow F = \int (2ty + 2y) \, dy
\]

\[
= (t+1)y^2 + B(t) = ty^2 + y^2 + B(t) \quad \text{(2)}
\]
Combine (1) and (2): 

$$F = \frac{1}{3} t^3 + ty^2 + y^2 \quad (+ \text{ const.})$$

Therefore

$$(t^2 + y^2) + 2(t+1)y \frac{dy}{dt} = 0$$

$$\Leftrightarrow \frac{d}{dt} \left( \frac{1}{3} t^3 + ty^2 + y^2 \right) = 0$$

$$\Leftrightarrow \frac{1}{3} t^3 + ty^2 + y^2 = \int 0 \, dt = c$$

Implicit solution is

$$\frac{1}{3} t^3 + ty^2 + y^2 = c, \quad c \text{arb.}$$

Convenient to find $c$ now, using IC

$$y(0) = -1$$

(could be done after solving for $y$)

$t=0, y=-1$:

$$\frac{1}{3} 0^3 + 0(-1)^2 + (-1)^2 = c$$

$$c = 1$$
Implicit soln. of IVP is
\[ \frac{1}{3} t^3 + ty^2 + y^2 = 1 \]

Explicit soln: solve for \( y \)
\[ y = \pm \sqrt{\frac{1 - \frac{1}{3} t^3}{t+1}} \]

Take - sign (why?)
\[ y(t) = - \sqrt{\frac{1 - \frac{1}{3} t^3}{t+1}} \]

Exercise: largest open interval in which soln. exists is \(-1 < t < 3^{1/3}\) (containing \( t = 0 \))

Example 1.8.B Solve
\[ \frac{x^2 + y^2}{x+1} + 2y \frac{dy}{dx} = 0 \]

\( M(x,y) \) \( N(x,y) \)

Not separable, not linear; exact?
\[ \frac{\partial M}{\partial y} = \frac{3}{2} y \left( \frac{x^2 + y^2}{x+1} \right) \neq \frac{2y}{x+1} \]
\[ \frac{\partial N}{\partial x} = \frac{3}{2} x (2y) = 0 \] (T.B.C.)

Not equal, ODE not exist.
1.8.2 Integrating factors

\[ M + N \frac{dy}{dx} = 0 \]

Suppose
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \]

so ODE is not exact. There is a chance that by multiplying by an integrating factor \( u(x,y) \), the resulting ODE would be exact:

\[ uM + uN \frac{dy}{dx} = 0 \]

Exact?

Need \( \frac{\partial}{\partial y}(uM) = \frac{\partial}{\partial x}(uN) \)

\[ \frac{\partial u}{\partial y} M + u \frac{\partial M}{\partial y} = \frac{\partial u}{\partial x} N + u \frac{\partial N}{\partial x} \]

Two special cases: i. \( u = u(x) \) only

i. If \( u = u(x) \) only then \( \frac{du}{dx} = u' \), \( \frac{du}{dy} = 0 \)

\[ \frac{\partial u}{\partial y} = u' N + u \frac{\partial N}{\partial x} \]
\[ u' = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} u \]

If \[ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \]

is a function of \( x \) only

then \( u(x) \) satisfies a linear ODE, can solve.

ii. Similarly, if \( u = u(y) \) only.