Example 2.3.B. A mass is projected upwards from the earth's surface with initial velocity \( v_0 \).

Considering gravity but neglecting friction, determine the value of \( v_0 \) needed to reach a maximum altitude \( h \). Also find the escape velocity.

Gravitational acceleration is proportional to the inverse square of the distance between the mass and the earth's centre with constant of proportionality \( gR^2 \), \( g \) = grav. accel. at earth's surface \(( \approx 9.8 \text{ m/s}^2 \) \)

\( R \) = earth's radius \(( \approx 6.4 \times 10^6 \text{ m} \)

Let \( x(t) = \) altitude at time \( t \)

\( x = \) upwards distance from Earth's surface

\( v = \frac{dx}{dt} = \) upwards velocity

\( a = \frac{dv}{dt} = \) upwards acceleration
\[
\frac{dv}{dt} = -gR^2 \frac{1}{(R+x)^2}
\]

Too many variables: \( v, x, t \)

Trick: think of \( v \) as function of \( x \):

Chain rule \( \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} \)

\[
v \frac{dv}{dx} = -gR^2 \frac{1}{(R+x)^2}
\]

\[
v \frac{dv}{dx} = -gR^2 \int \frac{1}{(R+x)^2} \, dx
\]

\[
\int uv \, du = -gR^2 \int \frac{1}{(x+R)^2} \, dx
\]
\[ \frac{1}{2}v^2 = gR^2 (x+R)^{-1} + c \]

Solve for \( c \) (don't have to solve for \( v \) first)

\[ v = v_0 \text{ when } x = 0 \]

\[ \frac{1}{2}v_0^2 = gR + c \]

so \( c = \frac{1}{2}v_0^2 - gR \)

and

\[ \frac{1}{2}v^2 = \frac{gR^2}{(x+R)} + \frac{1}{2}v_0^2 - gR \]

At max. altitude \( x=h \) : \( v(h) = 0 \)

\[ \frac{1}{2}0^2 = \frac{gR^2}{(h+R)} + \frac{1}{2}v_0^2 - gR \]

Solve for \( v_0 \):

\[ v_0 = \pm \sqrt{\frac{2gRh}{h+R}} \]

Take + sign ("physics")

\[ v_0 = \sqrt{\frac{2gRh}{h+R}} \]

initial vel. to reach max. alt. h
Escape velocity: \( v_0 \) when \( h \to \infty \)

\[
\begin{align*}
v_{\text{escape}} &= \lim_{h \to \infty} v_0 \\
&= \lim_{h \to \infty} \sqrt{\frac{2gRh}{h + R}} \\
&= \lim_{h \to \infty} \sqrt{\frac{2gR}{1 + R/h}} \\
&= \sqrt{2gR}
\end{align*}
\]

2.4 Differences between linear and nonlinear differential equations

Linear

\[
y' + p(t)y = g(t)
\]

Nonlinear General

\[
y' = f(t, y)
\]

Nonlinear or linear (typically nonlinear)
Theorem 2.4.1 [Existence and uniqueness for 1st-order linear equations]
If the functions \( p(t) \) and \( g(t) \) are continuous on an open interval \( I : \alpha < t < \beta \) containing the point \( t_0 \), then there exists a unique solution \( y = \phi(t) \) of the initial value problem
\[
y' + p(t)y = g(t), \quad y(t_0) = y_0 \quad (1), (2)
\]
defined for all \( \alpha < t < \beta \), for any prescribed initial value \( y_0 \).

Theorem 2.4.2 [Existence and uniqueness for 1st-order equations]
If the functions \( f(t,y) \) and \( \frac{df}{dy}(t,y) \) are continuous in an open rectangle \( \alpha < t < \beta, \gamma < y < \delta \) containing the point \( (t_0,y_0) \), then there exists a unique solution \( y = \phi(t) \) of the initial value problem
\[
y' = f(t,y), \quad y(t_0) = y_0 \quad (9)
\]
defined at least in some interval \( t_0 - h < t < t_0 + h \) that is contained in \( \alpha < t < \beta \).
Example 2.4, A

\[ ty' + 2y - 4t^2 = 0, \quad y(1) = 2 \]

Linear: both Theorems apply, but 2.4.1 gives more information.

Write as

\[ y' + \frac{2}{t} y = 4t, \quad y(1) = 2 \]

\[ p(t) = \frac{2}{t}, \quad g(t) = 4t \]

\[ p, g \text{ continuous on } -\infty < t < 0, \quad 0 < t < \infty \]

Only one of these intervals contains \( t_0 = 1 \). We know the solution is defined on \( 0 < t < \infty \).

Exercise: Solve and verify \( 0 < t < \infty \).
Example 2.4. B \quad y' - y^2 = 0 \quad y(0) = -2

Nonlinear: Thm. 2.4.1 does not apply

only 2.4.2 does

\[
y' = y^2, \quad y(0) = -2
\]

\[
f(t, y) \quad \frac{\partial f}{\partial y}(t, y) = 2y
\]

\(f, \frac{\partial f}{\partial y}\) are continuous on

\(-\infty < t < \infty, \quad -\infty < y < \infty\)

Solve:

\[
\frac{dy}{dt} = y^2
\]

\[
\frac{dy}{y^2} = dt
\]

\[
-\frac{1}{y} = t + c
\]

At \(t = 0, \quad y = -2\)

\[
\frac{1}{2} = 0 + c \quad , \quad c = \frac{1}{2}
\]
\[-\frac{1}{4}y = t + \frac{1}{2}\]
\[y = -\frac{1}{t + \frac{1}{2}}\]

\(t \neq -\frac{1}{2}\) so \(-\infty < t < -\frac{1}{2}\) or \(-\frac{1}{2} < t < \infty\)

The interval \(-\frac{1}{2} < t < \infty\) contains \(t_0 = 0\) so this is the interval on which the solution is defined:

\[y = y(t) = \phi(t) = -\frac{1}{t + \frac{1}{2}}, \quad -\frac{1}{2} < t < \infty\]
Exercise  Find interval of definition of soln. to
\[ y' = y^2, \quad y(0) = y_0 \]

i) \quad y_0 > 0
ii) \quad y_0 < 0
iii) \quad y_0 = 0

General solution

Linear eqns. have general solutions (formula with arb. constant that gives all solutions)
Nonlinear eqns. might not have a general solution.