Separable equs. cont.

Example 1.3. A Solve the IVP

\[ y' = xy, \quad y(0) = -1 \]

and determine the largest open interval in which the solution exists.

\[ \frac{dy}{dx} = xy \]

\[ \frac{1}{y} \, dy = x \, dx \]

\[ \int \frac{1}{y} \, dy = \int x \, dx \]

\[ \ln|y| = \frac{1}{2} x^2 + c \]

\[ |y| = e^{\frac{1}{2} x^2 + c} \]

\[ y = \pm e^c e^{\frac{1}{2} x^2} \]

At \( x = 0, \) \( y(0) = -1 \):

\[ -1 = \pm e^c e^{\frac{1}{2} \cdot 0^2} \]

\[ \text{take - sign} \]
\[-1 = -e^c \]
\[c = \ln(1) = 0\]

Solution of IVP is
\[y(x) = -e^{\frac{1}{2}x^2}\]

This is continuous for \(-\infty < x < +\infty\), so the largest open interval in which the solution exists is
\[-\infty < x < +\infty \quad \text{(or} \ (-\infty, \infty) \text{)} \]

Summary
\[y(x) = -e^{\frac{1}{2}x^2}, \quad -\infty < x < +\infty\]

Exercise Solve \[\frac{dy}{dx} = xy, \quad y(1) = 0\]

Method: Guess (look at slope field) and check

Example 1.3.B Solve the IVP
\[xy' + y^2 = 0, \quad y(-1) = 1,\]

and determine the largest open interval in which the solution exists.
\[ \frac{dy}{dx} = -\frac{y^2}{x} \]

\[-y^{-2}\, dy = \frac{1}{x}\, dx\]

\[\int (-y^{-2})\, dy = \int \frac{1}{x}\, dx\]

\[y^{-1} = \ln|\,x\,| + c \quad \text{(implicit soln.)}\]

\[y = \frac{1}{c + \ln|\,x\,|}\]

At \( x = -1, \quad y = 1 \):

\[1 = \frac{1}{c + \ln|\,-1\,|} = \frac{1}{c + \ln(1)} = 0\]

\[c = 1\]

\[y(x) = \frac{1}{1 + \ln|\,x\,|}\]

Note: \( x \neq 0, \quad 1 + \ln|\,x\,| \neq 0 \)

\[\ln|\,x\,| \neq -1\]

\[|\,x\,| \neq e^{-1}\]

\[x \neq \pm e^{-1} = \pm \frac{1}{e}\]

\( (e \approx 2.7) \)
Largest open interval in which solution exists is
\[-\infty < x < -e^{-1}\]

Graph of \( y = \frac{1}{1 + 2e|x|} \)

Solutions of ODEs must be continuous and solutions of IVP pass through a specified point

Solution of IVP

\[ y \neq y(x) = \frac{1}{1 + 2e|x|}, \quad -\infty < x < -e^{-1} \]
Example 1.3. C A satellite is sent up away from the earth's surface with initial velocity \( v_0 \). Considering gravity but neglecting friction, determine the value of \( v_0 \) required to for the satellite to reach a maximum altitude of \( H \). Also find the escape velocity ( \( H = \infty \) ).

Hint: acceleration due to gravity is proportional to the inverse square of the distance between the centres of the earth and satellite, with constant of proportionality \( g R^2 \),

\[
g = 9.8 \text{ m.s}^{-2} \quad \text{(accel. due to gravity at surface of earth)}
\]
\[
R = 6.4 \times 10^6 \text{ m} \quad \text{(radius of earth)}
\]

Let

\[
x = x(t) = \text{altitude of satellite (with satellite as point mass)}
\]
\[
v = \ddot{x} = \text{upwards velocity}
\]
\[
a = \dddot{x} = \text{upwards acceleration}
\]
Interpreting the hint

\[ a = -gR^2 \frac{1}{(R+x)^2} \]

\[ \frac{dv}{dt} \]

Trouble: three variables \( v, t, x \)

Think of \( v \) as a function of altitude \( x \)

\( v(x) \)

Chain Rule

\[ \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} \]

Now the DE becomes

\[ v \frac{dv}{dx} = -gR^2 \frac{1}{(R+x)^2} \]

IC is

\[ v(0) = v_0 \]

\[ x = 0 \]

Separable equ:

\[ v \, dv = -gR^2 \frac{1}{(R+x)^2} \, dx \]

\[ \int v \, dv = \int (-gR^2) \frac{1}{(R+x)^2} \, dx \]
\[ \frac{1}{2} u^2 = g R^2 (R + x)^{-1} + c \]