1. (a) \( y'' + 3y' + 2y = e^t. \)

The characteristic equation for the corresponding homogeneous equation \( y'' + 3y' + 2y = 0 \) is
\[
r^2 + 3r + 2 = (r + 1)(r + 2) = 0
\]
which has real distinct roots
\[
r_1 = -1, \quad r_2 = -2.
\]
The general solution of the corresponding homogeneous equation, or the complementary solution, is
\[
c_1 e^{-t} + c_2 e^{-2t},
\]
where \( c_1 \) and \( c_2 \) are arbitrary constants.

Since the corresponding homogeneous equation has constant coefficients, and the nonhomogeneous term \( g(t) = e^t \) appears in Table 3.5.1, we use the method of undetermined coefficients, and look for a particular solution in the form
\[
Y(t) = t^s A e^t.
\]
Trying \( s = 0 \) first, we see that \( A e^t \) is not a solution of the corresponding homogeneous equation, so the guess
\[
Y(t) = A e^t
\]
will work. To find the coefficient \( A \), we substitute our \( Y(t) \) into the nonhomogeneous differential equation to get
\[
(A + 3A + 2A)e^t = e^t,
\]
which can only be satisfied with \( A = \frac{1}{6} \), thus
\[
Y(t) = \frac{1}{6} e^t
\]
is a particular solution and the general solution to the nonhomogeneous equation is
\[
y(t) = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{6} e^t,
\]
where \( c_1, c_2 \) are arbitrary constants.

(b) \( y'' + 3y' + 2y = t e^t. \)

The corresponding homogeneous equation \( y'' + 3y' + 2y = 0 \) is the same as in (a), the complementary solution is
\[
c_1 e^{-t} + c_2 e^{-2t},
\]
where \( c_1 \) and \( c_2 \) are arbitrary constants.

We use the method of undetermined coefficients again, and look for a particular solution in the form
\[
Y(t) = t^s (At + B)e^{-t}.
\]
Trying $s = 0$ first, we see that one term of $Ate^{-t} + Be^{-t}$ is a solution of the corresponding homogeneous equation, so we should not use $s = 0$. Next trying $s = 1$, we see that no term of $At^2e^{-t} + Bte^{-t}$ is a solution of the corresponding homogeneous equation, so we use $s = 1$ and guess that

$$Y(t) = At^2e^{-t} + Bte^{-t}.$$  

We compute (using the product rule)

$$Y'(t) = -At^2e^{-t} + (2A - B)te^{-t} + Be^{-t}$$  

and

$$Y''(t) = At^2e^{-t} + (B - 4A)te^{-t} + (2A - 2B)e^{-t},$$

then substituting into the nonhomogeneous equation and cancelling the common factor of $e^{-t}$, we get

$$At^2 + (B - 4A)t + (2A - 2B) - 3At^2 + (6A - 3B)t + 3B + 2At^2 + 2Bt = t,$$

$$2At + (2A + B) = t.$$  

The coefficients of $t^2$ cancel, and matching the coefficients of $t$ and $t^0$, we get the two equations

$$2A = 1 \quad 2A + B = 0, $$

which has the solution

$$A = \frac{1}{2}, \quad B = -1,$$

so a particular solution is

$$Y(t) = \frac{1}{2} t^2e^{-t} - te^{-t}.$$  

The general solution of the nonhomogeneous equation is

$$y(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{2} t^2e^{-t} - te^{-t}.$$  

(c) \quad y'' + 3y' + 2y = e^t + t e^t.

The general solution to the nonhomogeneous equation is the sum of the general solution to the corresponding homogeneous equation and the particular solution of part (a) and the particular solution of part (b):

$$y(t) = c_1e^{-t} + c_2e^{-2t} + \frac{1}{6} e^t + \frac{1}{2} t^2e^{-t} - te^{-t}.$$  

d) \quad 4y'' + y = 2 \sec(t/2).

Although the corresponding homogeneous equation $4y'' + y = 0$ has constant coefficients, the form of the nonhomogeneous term $G(t) = 2 \sec(t/2)$ is not in Table 3.5.1, so we must use variation of parameters.

The characteristic equation of the corresponding homogeneous equation is

$$4r^2 + 1 = 0.$$
which has complex (purely imaginary) roots
\[ r_{1,2} = \pm i \frac{1}{2}, \]
so the complementary solution is
\[ c_1 y_1(t) + c_2 y_2(t) = c_1 \cos(t/2) + c_2 \sin(t/2). \]
To use variation of parameters, we write the nonhomogeneous equation in standard form, as
\[ y'' + \frac{1}{4} y = \frac{1}{2} \sec^2(t/2), \]
with nonhomogeneous term
\[ g(t) = \frac{1}{2} \sec^2(t/2). \]
Then we look for a solution of the nonhomogeneous equation in the form
\[ y(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \]
where \( u'_1(t) \) and \( u'_2(t) \) satisfy
\[ \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \]
which has the solution
\[ \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \frac{1}{W[y_1, y_2](t)} \begin{pmatrix} -y_2(t) g(t) \\ y_1(t) g(t) \end{pmatrix}. \]
Using \( g(t) = \frac{1}{2} \sec^2(t/2), y_1(t) = \cos(t/2), y_2(t) = \sin(t/2) \) and
\[ W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \frac{1}{2} \cos^2(t/2) + \frac{1}{2} \sin^2(t/2) = \frac{1}{2}, \]
and simplifying we have
\[ u'_1(t) = -\sin(t/2) \sec(t/2) = -\tan(t/2), \]
\[ u'_2(t) = \cos(t/2) \sec(t/2) = 1. \]
Then integrating (and ignoring constants of integration because we only need a particular solution) we get
\[ u_1(t) = -2 \ln |\sec(t/2)| = 2 \ln |\cos(t/2)|, \quad u_2(t) = t. \]
Therefore a particular solution of the nonhomogeneous equation is
\[ 2 \ln |\sec(t/2)| \cos(t/2) + t \sin(t/2), \]
and the general solution of the nonhomogeneous equation is
\[ y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) + 2 \ln |\sec(t/2)| \cos(t/2) + t \sin(t/2), \]
where \( c_1, c_2 \) are arbitrary constants.
2. (a) \( y'' + 3y' + 2y = e^{2t}(t + 1)\sin(5t) \).

Recall in 1(a) we have already found the general solution of the corresponding homogeneous equation, as

\[ c_1e^{-t} + c_2e^{-2t} . \]

According to Table 3.5.1, for the method of undetermined coefficients with the nonhomogeneous term \( g(t) = e^{2t}(t + 1)\sin(5t) = (t^2 + t + 0)e^{-2t}\sin(5t) \), we should try a particular solution of the form

\[ t^s \left[ (A_0t^2 + A_1t + A_2)e^{-2t}\cos(5t) + (B_0t^2 + B_1t + B_2)e^{-2t}\sin(5t) \right] . \]

Trying \( s = 0 \) first, we see that no term of the trial solution is a solution of the corresponding homogeneous equation, so the correct guess for a particular solution is

\[ Y(t) = (A_0t^2 + A_1t + A_2)e^{-2t}\cos(5t) + (B_0t^2 + B_1t + B_2)e^{-2t}\sin(5t) . \]

(b) \( y'' + 16y = t^2\cos(4t) + t\sin(4t) + \sin(5t) \).

The characteristic equation of the corresponding homogeneous equation is \( r^2 + 16 = 0 \), which has complex (purely imaginary) roots \( r_{1,2} = \pm 4i \), so the complementary solution (general solution of the corresponding homogeneous equation) is

\[ c_1\cos(4t) + c_2\sin(4t) . \]

According to the table, we guess a particular solution of the nonhomogeneous equation in the form

\[ t^s \left[ (A_0t^2 + A_1t + A_2)\cos(4t) + (B_0t^2 + B_1t + B_2)\sin(4t) \right] + C\cos(5t) + D\sin(5t) , \]

but trying \( s = 0 \) we see that the terms \( A_2\cos(4t) \) and \( B_2\sin(4t) \) are solutions of the corresponding homogeneous equation, so we should not take \( s = 0 \). Then trying \( s = 1 \) we see that no terms of the trial particular solution are solutions of the corresponding homogeneous equation, so the correct guess for a particular solution is

\[ Y(t) = t \left[ (A_0t^2 + A_1t + A_2)\cos(4t) + (B_0t^2 + B_1t + B_2)\sin(4t) \right] + C\cos(5t) + D\sin(5t) . \]

3. (a) \( 2ty'' + (1 - 4t)y' + (2t - 1)y = 0, 0 < t < \infty \).

If \( y(t) = e^t \), then \( y'(t) = e^t \) and \( y''(t) = e^t \), and substituting into the homogeneous equation we get

\[ 2t(e^t) + (1 - 4t)(e^t) + (2t - 1)(e^t) = [(2t - 4t + 2t) + (1 - 1)]e^t = 0, \]

which verifies that \( y(t) = e^t \) is a solution.

(b) \( 2ty'' + (1 - 4t)y' + (2t - 1)y = e^t, 0 < t < \infty \). We first need to find the general solution of the corresponding homogeneous equation, which was considered in part (a). We have already checked that \( y_1(t) = e^t \) is a solution of that equation, so we use the method of reduction of order, to find another solution in the form

\[ y(t) = v(t)y_1(t) = v(t)e^t . \]
We compute

\[ y' = v' y_1 + vy_1', \quad y'' = v'' y_1 + 2v'y_1' + vy_1'', \]

and substitute into the corresponding homogeneous equation

\[ 2ty'' + (1 - 4t)y' + (2t - 1)y = 0 \]

to obtain

\[
2t(v''y_1 + 2v'y_1' + vy_1'') + (1 - 4t)(v'y_1 + vy_1') + (2t - 1)vy_1 = 0
\]

which is a linear 1st-order equation for \( v' \). The integrating factor for this last equation is

\[ \mu(t) = e^{\int \frac{1}{2t} \, dt} = e^{(1/2)\ln(t)} = \sqrt{t}, \]

so multiplying by \( \mu(t) \) and identifying the left hand side as the derivative of a product, we have

\[
(\sqrt{t} v')' = 0
\]

\[
\sqrt{t} v' = c_2
\]

\[
v' = c_2 t^{-1/2}
\]

\[
v(t) = 2c_2 t^{1/2} + c_1,
\]

where \( c_1, c_2 \) are arbitrary constants. To get a second solution \( y_2(t) \) of the corresponding homogeneous equation that is not a constant multiple of \( y_1(t) \), we take \( c_1 = 0 \) and \( c_2 = \frac{1}{2} \), for example, so we choose

\[ y_2(t) = \sqrt{t} e^t. \]

We check

\[ W[y_1, y_2](t) = y_1(t)y_2'(t) - y_2(t)y_1'(t) = \frac{e^{2t}}{2\sqrt{t}}, \]

which is nonzero for all \( 0 < t < \infty \), so \( y_1(t) = e^t \) and \( y_2(t) = \sqrt{t} e^t \) form a fundamental set of solutions of the corresponding homogeneous equation on the interval \( 0 < t < \infty \), and the complementary solution is

\[ c_1 e^t + c_2 \sqrt{t} e^t \quad (0 < t < \infty). \]

Now we use variation of parameters to find a particular solution of the nonhomogeneous equation, as

\[ Y(t) = u_1(t)e^t + u_2(t)\sqrt{t} e^t, \]

where the nonhomogeneous equation is written in standard form

\[ y'' + \left( \frac{1}{2t} - 2 \right) y' + \left( 1 - \frac{1}{2t} \right) y = \frac{e^t}{2t}, \]

with the nonhomogeneous term

\[ g(t) = \frac{e^t}{2t}. \]
Then

\[ u_1(t) = - \int \frac{y_2 g}{W[y_1, y_2]} = - \int \frac{2\sqrt{t}}{e^{2t}} \sqrt{t}e^t \frac{e^t}{2t} dt = - \int 1 dt = -t, \]

and

\[ u_2(t) = \int \frac{y_1 g}{W[y_1, y_2]} = \int \frac{2\sqrt{t}}{e^{2t}} \sqrt{t}e^t \frac{e^t}{2t} dt = \int \frac{1}{\sqrt{t}} dt = 2\sqrt{t}, \]

(ignoring constants of integration) so a particular solution is

\[ Y(t) = (-t)e^t + (2\sqrt{t})e^t = te^t \]

(which can now be easily checked), and finally the general solution of the nonhomogeneous equation is

\[ y(t) = c_1 e^t + c_2 \sqrt{t}e^t + te^t, \]

where \( c_1, c_2 \) are arbitrary constants.

4. \( 4y'' + y = g(t), \quad y(0) = -1, \quad y'(0) = 1. \)

Even though the corresponding homogeneous equation \( 4y'' + y = 0 \) has constant coefficients, we use the method of variation of parameters because the nonhomogeneous term is an arbitrary function.

As in 1(d), the characteristic equation for the corresponding homogeneous equation is

\[ 4r^2 + 1 = 0, \]

which has complex (in fact, purely imaginary) roots \( r_1 = i \frac{1}{2}, \quad r_2 = -i \frac{1}{2} \), and the general solution of the corresponding homogeneous equation is

\[ c_1 y_1(t) + c_2 y_2(t) = c_1 \cos(t/2) + c_2 \sin(t/2). \]

The nonhomogeneous equation should first be written in standard form as

\[ y'' + \frac{1}{4} y = \frac{1}{4} g(t). \]

Then by the method of variation of parameters, a particular solution is

\[ Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t) \cos(t/2) + u_2(t) \sin(t/2). \]

We have

\[ u'_1(t) = - \frac{y_2(t)}{W[y_1, y_2](t)} \frac{\frac{1}{2} g(t)}{W[y_1, y_2](t)}, \quad u'_2(t) = \frac{y_1(t)}{W[y_1, y_2](t)} \frac{\frac{1}{2} g(t)}{W[y_1, y_2](t)} \]

where (from 1(d))

\[ y_1(t) = \cos(t/2), \quad y_2(t) = \sin(t/2), \quad W[y_1, y_2](t) = \frac{1}{2}, \]

so we take

\[ u_1(t) = -\frac{1}{2} \int_0^t g(s) \sin(s/2) ds, \quad u_2(t) = \frac{1}{2} \int_0^t g(s) \cos(s/2) ds, \]
and the general solution of the nonhomogeneous equation is

\[ y(t) = c_1 \cos(t/2) + c_2 \sin(t/2) - \frac{1}{2} \cos(t/2) \int_0^t g(s) \sin(s/2) \, ds + \frac{1}{2} \sin(t/2) \int_0^t g(s) \cos(s/2) \, ds, \]

where \( c_1, c_2 \) are arbitrary constants.

Now we choose \( c_1, c_2 \) to satisfy the initial conditions \( y(0) = -1 \) and \( y'(0) = 1 \). Differentiating, we use the product rule and the fundamental theorem of calculus to get

\[ y'(t) = -\frac{1}{2} c_1 \sin(t/2) + \frac{1}{2} c_2 \cos(t/2) + \frac{1}{4} \sin(t/2) \int_0^t g(s) \sin(s/2) \, ds - \frac{1}{2} \cos(t/2) g(t) \sin(t/2) + \frac{1}{4} \cos(t/2) \int_0^t g(s) \cos(s/2) \, ds + \frac{1}{2} \sin(t/2) g(t) \cos(t/2). \]

Evaluating the expressions for \( y(t) \) and \( y'(t) \) at \( t = 0 \) (the integrals all vanish), and applying the initial conditions, we get

\[ y(0) = c_1 = -1, \quad y'(0) = \frac{1}{2} c_2 = 1, \]

thus

\[ c_1 = -1, \quad c_2 = 2, \]

and the solution of the initial value problem is

\[ y(t) = -\cos(t/2) + 2 \sin(t/2) - \frac{1}{2} \cos(t/2) \int_0^t g(s) \sin(s/2) \, ds + \frac{1}{2} \sin(t/2) \int_0^t g(s) \cos(s/2) \, ds \]

\[ = -\cos(t/2) + 2 \sin(t/2) + \int_0^t \frac{1}{2} [\sin(t/2) \cos(s/2) - \cos(t/2) \sin(s/2)] g(s) \, ds \]

\[ = -\cos(t/2) + 2 \sin(t/2) + \int_0^t \frac{1}{2} \sin((t - s)/2) g(s) \, ds. \]

(The last integral is called a convolution.)