1. \( y'' - 5y' + 4y = 0, \ y(0) = 2, \ y'(0) = -3. \)

The characteristic equation is

\[ r^2 - 5r + 4 = (r - 4)(r - 1) = 0 \]

which has real distinct roots

\[ r_1 = 4, \quad r_2 = 1. \]

The general solution is

\[ y(t) = c_1 e^{4t} + c_2 e^t, \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. To satisfy the initial conditions we need to choose \( c_1, \ c_2 \) so that

\[ y(0) = c_1 + c_2 = 2, \quad y'(0) = 4c_1 + c_2 = -3, \]

therefore

\[ c_1 = -\frac{5}{3}, \quad c_2 = \frac{11}{3}, \]

and the solution to the initial value problem is

\[ y(t) = -\frac{5}{3} e^{4t} + \frac{11}{3} e^t. \]

2. \( t^2 y'' - 2y = 0, \ y(-1) = -1, \ y'(-1) = -4. \)

(a) In standard form, the differential equation is

\[ y'' - \frac{2}{t^2} y = 0, \]

and the functions \( p(t) = 0, \ q(t) = -2/t^2, \ g(t) = 0 \) are all continuous on either of the open intervals \(-\infty < t < 0 \) or \( 0 < t < +\infty \). The interval must contain the initial time \( t_0 = -1 \), so by Theorem 3.2.1 the unique solution \( y = \phi(t) \) is defined on the open interval

\[ -\infty < t < 0. \]

(b) To verify that a given function is a solution, we differentiate it and see if it and its derivative(s) satisfy the differential equation. Differentiating \( y_1(t) = t^2 \) twice, we get \( y_1'(t) = 2t, \ y_1''(t) = 2 \), then \( t^2 y_1''(t) - 2y_1(t) = t^2 \cdot 2 - 2 \cdot 2 = 0 \), which verifies that \( y_1(t) = t^2 \) is a solution.

Doing the same for \( y_2(t) = 1/t = t^{-1} \), we get \( y_2'(t) = -1/t^2, \ y_2''(t) = 2/t^3 \), then \( t^2 y_2''(t) - 2y_2(t) = t^2 (2/t^3) - 2(1/t) = (2/t) - (2/t) = 0 \), which verifies that \( y_2''(t) = 1/t \) is a solution.

Since \( y_2(t) = 1/t \) is discontinuous at \( t = 0 \), **both** \( y_1(t) \) and \( y_2(t) \) are solutions on either \(-\infty < t < 0 \) or \( 0 < t < \infty \).
(c) We already know $y_1(t)$ and $y_2(t)$ are solutions, it remains to verify that the Wronskian does not vanish:

$$W[y_1, y_2](t) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = (t^2)(-t^{-2}) - (t^{-1})(2t) = -3$$

which is nonzero for all $t$. So $y_1(t)$ is a solution, $y_2(t)$ is a solution, and $W[y_1, y_2](t) \neq 0$ are all true simultaneously on the two intervals

$$-\infty < t < 0, \quad \text{or} \quad 0 < t < \infty.$$

(d) By the results of parts (b) and (c), and the theory of second-order linear homogeneous differential equations, the general solution is

$$y(t) = c_1 t^2 + c_2 \frac{1}{t},$$

where $c_1$, $c_2$ are arbitrary constants. The initial conditions require

$$y(-1) = c_1 - c_2 = -1, \quad y'(-1) = -2c_1 - c_2 = -4,$$

therefore

$$c_1 = 1, \quad c_2 = 2,$$

and the solution to the initial value problem is

$$y = \phi(t) = t^2 + 2, \quad -\infty < t < 0.$$

3. $y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -1.$

The characteristic equation is

$$r^2 + 2r + 2 = 0,$$

which has complex roots

$$r_1 = -1 + i, \quad r_2 = -1 - i,$$

and the general solution is

$$y(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t,$$

where $c_1$, $c_2$ are arbitrary constants. Also, $y'(t) = c_1 (-e^{-t} \cos t - e^{-t} \sin t) + c_2 (-e^{-t} \sin t + e^{-t} \cos t)$.

Now we find $c_1$ and $c_2$ to satisfy the initial condition: evaluating the expressions above at $t = 0$, we get $y(0) = c_1$, $y'(0) = -c_1 + c_2$, therefore we solve

$$c_1 = 1, \quad -c_1 + c_2 = -1,$$

and get

$$c_1 = 1, \quad c_2 = 0.$$
The solution to the initial value problem is
\[ y(t) = e^{-t} \cos t. \]

4. \( 4y'' + 4y' + y = 0, \quad y(2) = -6, \quad y'(2) = 4. \)

The characteristic equation is
\[ 4r^2 + 4r + 1 = 0, \]
which has repeated roots
\[ r_1 = -\frac{1}{2}, \quad r_2 = -\frac{1}{2}, \]
and the general solution is
\[ y(t) = c_1 e^{-t/2} + c_2 te^{-t/2}, \]
where \( c_1, c_2 \) are arbitrary constants. Also, \( y'(t) = -\frac{1}{2}c_1 e^{-t/2} + c_2 (1 - \frac{1}{2}t) e^{-t/2}. \)

Now we find \( c_1 \) and \( c_2 \) to satisfy the initial condition: evaluating the expressions above at \( t = 2 \), we get \( y(2) = c_1 e^{-1} + c_2 2e^{-1}, \ y'(2) = c_1 \left(-\frac{1}{2}e^{-1}\right), \) therefore we must solve
\[ e^{-1}c_1 + 2e^{-1} = -6, \quad -\frac{1}{2}e^{-1}c_1 = -4, \]
and get
\[ c_1 = -8e, \quad c_2 = e. \]
The solution to the initial value problem is
\[ y(t) = -8e \cdot e^{-t/2} + e \cdot te^{-t/2}. \]
(To make the algebra a little easier, we could write the general solution in terms of \( t - 2 \) instead of \( t \), as
\[ y(t) = k_1 e^{-(t-2)/2} + k_2 (t - 2)e^{-(t-2)/2} \]
and solve more easily for \( k_1 = -6, \quad k_2 = 1, \) thus
\[ y(t) = -6e^{-(t-2)/2} + (t - 2)e^{-(t-2)/2}, \]
which is the same as above.)

5. \( t^2 y'' + 9ty' + 16y = 0, \quad y(1) = -1, \quad y'(1) = 1. \)

(a) In standard form, the differential equation is
\[ y'' + \frac{9}{t} y' + \frac{16}{t^2} y = 0, \]
and the functions \( p(t) = 9/t, \quad q(t) = 16/t^2, \quad g(t) = 0 \) are all continuous on either of the open intervals \(-\infty < t < 0\) or \( 0 < t < +\infty\). The interval must contain the initial time \( t_0 = 1 \), so by Theorem 3.2.1 the unique solution \( y = \phi(t) \) is defined on the open interval
\[ 0 < t < \infty. \]
(b) Differentiating \( y_1(t) = t^{-4} \) twice, we get \( y_1'(t) = -4t^{-5}, \ y_1''(t) = 20t^{-6} \), then \( t^2y_1''(t) + 9ty_1'(t) + 16y_1(t) = t^2 \cdot 20t^{-6} + 9t(-4t^{-5}) + 16 \cdot t^{-4} = (20 - 36 + 16)t^{-4} = 0 \), which verifies that \( y_1(t) = t^{-4} \) is a solution.

To find another solution using the method of reduction of order, we put \( y = v(t)y_1(t) = v(t)t^{-4} \). Then we calculate \( y' = v't^{-4} - 4vt^{-5}, \ y'' = v''t^{-4} - 8vt^{-5} + 20vt^{-6} \) and substitute these expressions into the left-hand side of the differential equation:

\[
t^2(v''t^{-4} - 8vt^{-5} + 20vt^{-6}) + 9t(v't^{-4} - 4vt^{-5}) + 16(vt^{-4}) = 0.
\]

After simplifying, we get

\[
t^{-2}v'' + t^{-3}v' = 0
\]

(the coefficient of \( v \) should be 0). Multiplying by \( t^2 \), we get

\[
(v')' + \frac{1}{t} (v') = 0.
\]

This is a first-order linear equation for \( v' \). We may solve for \( v' \) by the method of integrating factors, or by separating variables. For example, multiplying by the integrating factor \( \mu(t) = e^{\int (1/t) dt} = e^\ln t = t \) (taking \( t > 0 \) because of the initial time \( t_0 = 1 \)), we get

\[
t(v')' + v' = 0
\]

\[
(tv')' = 0
\]

\[
tv' = c_2
\]

\[
v'(t) = \frac{c_2}{t},
\]

where \( c_2 \) is an arbitrary constant. Then integrate to get

\[
v(t) = \int \frac{c_2}{t} \, dt = c_1 + c_2 \ln |t| \quad (t \neq 0),
\]

where \( c_1 \) is also an arbitrary constant.

Then \( y(t) = v(t)y_1(t) = (c_1 + c_2 \ln |t|)t^{-4} \) is a solution for any \( c_1, c_2 \). Taking \( c_1 = 0 \) and \( c_2 = 1 \), we choose the solution

\[
y_2(t) = t^{-4} \ln |t|,
\]

which is not equal to a constant multiple of \( y_1(t) = t^{-4} \).

(c) To verify that \( y_1 \) and \( y_2 \) form a fundamental set of solutions, we note we have shown above that they are both solutions of the differential equation, on \(-\infty < t < 0 \) or on \( 0 < t < \infty \) so it remains to compute the Wronskian

\[
W[y_1, y_2](t) = \begin{vmatrix}
t^{-4} & t^{-4} \ln |t| \\
-4t^{-5} & -4t^{-5} \ln |t| + t^{-5}
\end{vmatrix} = t^{-9} = \frac{1}{t^9},
\]

which we verify is never zero on \( 0 < t < \infty \), or on \(-\infty < t < 0 \). So \( y_1 \) and \( y_2 \) form a fundamental set of solutions on

\[-\infty < t < 0 \quad \text{or} \quad 0 < t < \infty\]
(d) To solve the initial value problem we write down the general solution

\[ y(t) = c_1 t^{-4} + c_2 t^{-4} \ln t, \]

(we use \( t > 0 \) because of the initial condition) and also compute its derivative \( y'(t) = -4c_1 t^{-5} + c_2 (t^{-5} - 4t^{-5} \ln t) \), then evaluate at \( t = 1 \) to get \( y(1) = c_1, \ y'(1) = -4c_1 + c_2 \). So we set

\[ c_1 = -1, \quad -4c_1 + c_2 = 1 \]

in order to satisfy the initial conditions, therefore

\[ c_1 = -1, \quad c_2 = -3, \]

and the solution to the initial value problem is

\[ y = \phi(t) = -t^{-4} - 3t^{-4} \ln t, \quad 0 < t < \infty. \]