1. To verify that a given function is a solution, we check that it is continuous, we differentiate it, substitute it into the differential equation and see if the differential equation is satisfied. We also substitute \( t = t_0 \) into the function see if it satisfies the initial condition.

Note that \( e^{t^2} \) and \( \int_0^t e^{-u^2} \, du \) are both continuous functions of \( t \), so \( y(t) \) is continuous (for all \( t, -\infty < t < \infty \)). Differentiating the expression for \( y(t) \), using the Product Rule and the Fundamental Theorem of Calculus, we get

\[
y'(t) = 2te^{t^2} \int_0^t e^{-u^2} \, du + e^{t^2} e^{-t^2} + 2te^{t^2}
\]

\[
= 2t \left( e^{t^2} \int_0^t e^{-u^2} \, du + e^{t^2} \right) + 1
\]

\[
= 2ty(t) + 1,
\]

which verifies that the differential equation \( y' - 2ty = 1 \) is satisfied.

Evaluating the expression at \( t = 0 \) gives

\[
y(0) = e^0 \cdot 0 + e^0 = 1,
\]

which verifies that the initial condition is satisfied.

2. (a) \( y' + 2y = te^{-2t}, \quad y(1) = 0. \)

The first-order differential equation is linear. The integrating factor \( \mu(t) \) needs to satisfy \( \mu' = 2\mu \), from which we recognize the solution is \( \mu(t) = e^{\int 2 \, dx} = e^{2t+k} \), \( k \) arbitrary. We can take \( k = 1 \) here, and multiply the DE by the integrating factor

\[
\mu(t) = e^{2t}
\]

to get

\[
e^{2t}y' + 2e^{2t}y = t,
\]

\[
\left( e^{2t}y \right)' = t,
\]

\[
e^{2t}y = \int t \, dt,
\]

\[
e^{2t}y = \frac{1}{2}t^2 + c,
\]

where \( c \) is an arbitrary constant.

Multiply by \( e^{-2t} \) to get the general solution

\[
y(t) = \frac{1}{2}t^2 e^{-2t} + c e^{-2t},
\]

and now use the initial condition at \( t = 1, y(1) = 0 \) to get

\[
\frac{1}{2}e^{-2} + c e^{-2} = 0
\]

1
and solve for 

\[ c = -\frac{1}{2}. \]

The solution to the IVP is 

\[ y(t) = \frac{1}{2} t^2 e^{-t^2} - \frac{1}{2} e^{-2t}, \]

which is defined in the interval 

\[ -\infty < t < \infty. \]

(b) \( y' + (2/x)y = x^{-2}, \ x > 0. \)

This is a linear first-order equation. The integrating factor \( \mu(x) \) needs to satisfy \( \mu' = (2/x)\mu \), which we solve by writing \( d\mu/\mu = 2/x \) and integrating to get \( \ln|\mu| = 2\ln x + k \) \( (x > 0) \), \( \mu(x) = e^{\int (2/x)dx} = e^{2\ln x + k} = e^k x^2 \), \( k \) arbitrary. We can take \( k = 0 \) here and an integrating factor is 

\[ \mu(x) = x^2. \]

Multiply the differential equation by \( \mu(x) = x^2 \) and get 

\[ x^2 y' + 2xy = 1, \]

\[ (x^2 y)' = 1, \]

\[ x^2 y = \int 1 \, dx, \]

\[ x^2 y = x + c, \]

where \( c \) is an arbitrary constant.

Divide by \( x^2 \) to get the general solution:

\[ y(t) = \frac{1}{x} + \frac{c}{x^2}, \quad x > 0, \]

where \( c \) is an arbitrary constant.

(c) \( y' = x^2/y(1 + x^3). \)

This is a nonlinear first-order equation, and it is separable. Write the DE as 

\[ dy/dx = x^2/y(1 + x^3) \]

and separate variables to get 

\[ y \, dy = x^2 \frac{dx}{1 + x^3}. \]

Integrate both sides to get 

\[ \frac{1}{2} y^2 = \frac{1}{3} \ln|1 + x^3| + c, \]

and solve for \( y \) to get 

\[ y(x) = \pm \sqrt{\frac{2}{3} \ln|1 + x^3| + c}, \]

where \( c \) is an arbitrary constant.

(d) \( y' = (1 - 2x)y^2, \ y(0) = -1/6. \)

The first-order equation is nonlinear and separable. Write it as 

\[ -y^{-2} \, dy = (2x - 1) \, dx, \]
then integrate to obtain

\[ y^{-1} = x^2 - x + c, \]

where \( c \) is an arbitrary constant. Substituting \( x = 0 \) and \( y = -1/6 \), we find that

\[ c = -6, \]

then \( y^{-1} = x^2 - x - 6 \), or in explicit form

\[ y(x) = \frac{1}{x^2 - x - 6}. \]

Noting that \( x^2 - x - 6 = (x + 2)(x - 3) \), we see that for \( y(x) \) to be defined, we need \( x \neq -2 \) and \( x \neq 3 \). Then \( y(x) \) might be defined for \( -\infty < x < -2 \) or \( -2 < x < 3 \) or \( 3 < x < \infty \). The interval in which the solution is defined must contain \( x = 0 \), so the interval is

\[ -2 < x < 3. \]

3. \( ay' + by = 0 \).

Letting \( y = e^{rt} \), we calculate \( y' = re^{rt} \) and substituting into the differential equation gives

\[ a( re^{rt}) + b(e^{rt}) = 0. \]

Dividing by \( e^{rt} \) (which is never zero) gives the algebraic equation

\[ ar + b = 0. \]

This is easily solved to give

\[ r = -\frac{b}{a}. \]

Now verify that \( y(t) = e^{-(b/a)t} \) is indeed a solution, by substituting it into the differential equation:

\[ a \left( -\frac{b}{a} e^{-bt/a} \right) + b \left( e^{-bt/a} \right) = -be^{-bt/a} + be^{-bt/a} = 0, \]

as is required for a solution.

4. Let \( M(t) \) be the amount of drug, in mg, present in the bloodstream at time \( t \) in hr.

(a) The drug enters the bloodstream at a rate in of

\[ (5 \text{ mg/cm}^3)(100 \text{ cm}^3/\text{hr}) = 500 \text{ mg/hr}, \]

and leaves the bloodstream at a rate out (the amount present at any time \( t \) is \( M(t) \) mg) of

\[ (0.4 \text{ hr}^{-1})(M \text{ mg}) = 0.4 M \text{ mg/hr}. \]

Therefore the total rate the drug enters the bloodstream is given by the differential equation

\[ \frac{dM}{dt} = \text{rate in} - \text{rate out} \]

\[ = 500 - 0.4 M, \]
and the units are \( \text{mg/hr} \).

(b) We interpret “after a long time” as the limiting value of \( M(t) \) when \( t \to \infty \). This can be found \textit{without} explicitly solving for \( M(t) \) by looking for an \textit{equilibrium solution}. Setting \( dM/dt = 0 \) in the differential equation and solving for \( M \) gives

\[
M = \frac{500}{0.4} = 1250 \text{ mg}.
\]

Alternatively, you could do part (c) first, then take the limit as \( t \to \infty \).

(c) We assume that when the intravenous unit is first hooked up to the patient, there is no drug in the bloodstream, so the initial condition is

\[
M(0) = 0.
\]

To solve the DE, we recognize it as a linear first-order equation and write it as

\[
M' + 0.4M = 500.
\]

An integrating factor is

\[
\mu(t) = e^{0.4t},
\]

multiplying the DE by this integrating factor gives

\[
(e^{0.4t}M)' = 500e^{0.4t},
\]

then integrating gives

\[
e^{0.4t}M = 1250e^{0.4t} + c,
\]

where \( c \) is an arbitrary constant. Using the initial condition \( M(0) = 0 \) we get \( c = -1250 \) (notice it is a little faster to find \( c \) if we don’t solve for \( M \) first) and so the solution of the initial value problem, which is the amount, in mg, of the drug that is present in the bloodstream at time \( t \), in hrs:

\[
M(t) = 1250 - 1250e^{-0.4t}.
\]

Notice that we have \( \lim_{t \to \infty} M(t) = 1250 \), as predicted in part (b).