1. (a) Sketch graph of \( r = 1 + \cos (3\theta) \) in polar coordinates. 
(b) Find the area of the region enclosed by that graph. 

\[ A = \int_0^{2\pi} \int_0^{1+\cos (3\theta)} r\, dr\, d\theta \]
\[ = \frac{1}{2} \int_0^{2\pi} (1 + \cos (3\theta))^2 \, d\theta \]
\[ = \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos (3\theta) + \cos^2 (3\theta)) \, d\theta \]
\[ = \frac{1}{2} (2\pi + \pi) \]
\[ = \frac{3\pi}{2} \]

2. (a) Evaluate the integral \( \int \int_D \sin (x^2 + y^2) \, dA \) where \( D \) is the unit circle \( x^2 + y^2 \leq 1 \).
(b) Evaluate the integral \( \int \int_D \sin (4x^2 + y^2) \, dA \) where \( D \) is an ellipse \( x^2 + (y/2)^2 \leq 1 \).

\[ I = \int_0^{2\pi} \int_0^{1} \sin (r^2) \, r\, dr\, d\theta \]
\[ = 2\pi \left( \frac{-\cos (r^2)}{2} \right)_0^1 = \pi (1 - \cos 1) \]

(b) Solution 1:
\[ \int \int_D \sin (4x^2 + y^2) \, dA = 4 \int_0^{1} \int_0^{\sqrt{1-x^2}} \sin (4x^2 + y^2) \, dy\, dx \]
\[ (\text{change var } y = 2\hat{y}) = 4 \int_0^{1} \int_0^{\sqrt{1-x^2}} \sin (4x^2 + 4\hat{y}^2) \, 2d\hat{y}\, dx \]
\[ = \left( \frac{\pi}{2} \right) \left( \int_0^{\pi/2} \int_0^1 \sin (4r^2) \, r\, dr\, d\theta \right) \]
\[ = \frac{\pi}{2} (1 - \cos 4) \]

Solution 2: [This solution requires a general change of variables which we have not covered yet]. 
The ellipse can be parametrized as \( x = r \cos \theta, \ y = 2r \sin \theta, \) with \( r = 0 \ldots 1, \ \theta = 0 \ldots 2\pi. \) Then it can be shown (see general change of variables) that \( dx\, dy = 2r\, dr\, d\theta. \) So we get
\[ \int \int_D \sin (4x^2 + y^2) \, dA = \int_0^{2\pi} \int_0^{1} \sin (4r^2) \, 2r\, dr\, d\theta = \frac{\pi}{2} (1 - \cos 4). \]
3. (a) Sketch the curves \( r = 1 \) and \( r = 2 + \cos 4\theta \) in polar coordinates.

(b) Determine the area of the region that is inbetween these two curves.

(c) What is the mass of the lamellar region in part (b) if its density is given by \( \rho(r, \theta) = r \)?

Answer. (a)

(b)

\[
A = \int_{0}^{2\pi} \int_{0}^{2 + \cos(4\theta)} r \, dr \, d\theta - \pi \\
= \frac{1}{2} \int_{0}^{2\pi} (2 + \cos (4\theta))^2 \, d\theta - \pi \\
= \frac{1}{2} \int_{0}^{2\pi} (4 + 4 \cos (4\theta) + \cos^2 (4\theta)) \, d\theta - \pi \\
= \frac{1}{2} (2\pi \cdot 4 + \pi) - \pi \\
= \frac{7\pi}{2}
\]

(c)

\[
m = \int_{0}^{2\pi} \int_{1}^{2 + \cos(4\theta)} r^2 \, dr \, d\theta \\
= \frac{1}{3} \int_{0}^{2\pi} \left[ (2 + \cos (4\theta))^3 - 1 \right] \, d\theta \\
= \frac{1}{3} \int_{0}^{2\pi} [8 + 6 \cos^2 (4\theta) - 1] \, d\theta \quad \text{(odd powers of cos integrate to zero)} \\
= \frac{20\pi}{3}.
\]

4. Consider the following region consisting of a triangle and a square:

The square has density \( \rho(x, y) = 1 \) and the triangle has density \( \rho(x, y) = x^2 \).

(a) Find the mass of the entire region.
(b) Find the center of mass of this region.

**Answer.**  
(a) The mass of the square is \(2 \times 1 \times 1 = 2\). The mass of the triangle is given by

\[
2 \int_0^1 \int_1^{2-x} x^2 \, dy \, dx = 2 \int_0^1 x^2 (1-x) \, dx = 2 \left( \frac{1}{3} - \frac{1}{4} \right) = \frac{1}{6}
\]

So the total mass is \(2 + \frac{1}{6} = \frac{13}{6}\).

(b) The x-moment of mass, \(M_y\), is 0 by symmetry. For the y-moment \(M_x\), we compute

\[
M_x = \bar{y}_{\text{square}} m_{\text{square}} = \frac{1}{2} \cdot 2 = 1.
\]

and

\[
M_{x,\text{triangle}} = 2 \int_0^1 \int_1^{2-x} yx^2 \, dy \, dx
\]

\[= \int_0^1 x^2 (2-x)^2 - 1 \, dx\]

\[= \int_0^1 (x^3 - 4x^3 + 3x^2) \, dx\]

\[= \frac{1}{5}\]

so that in total,

\[M_x = M_{x,\text{square}} + M_{x,\text{triangle}} = 1 + \frac{1}{5}\]

So we get:

\[
\bar{x} = 0, \quad \bar{y} = \frac{1 + \frac{1}{5}}{2 + \frac{1}{6}} \approx 0.5538.
\]

5. (a) Sketch the graphs of \(r = 1\) and \(r = 2 \cos \theta\) for \(\theta \in (-\pi/2, \pi/2)\). Identify the intersection points of these two circles.

(b) Find the area of the region that lies inside circle \(r = 2 \cos \theta\) and outside the circle \(r = 1\).

(c) Find the x,y-moments and the center of mass of the region in part (b). Please note: You may use the following facts without proofs:

\[
\int_0^{\pi/3} \cos^2(\theta) \, d\theta = \frac{\sqrt{3}}{8} + \frac{\pi}{6}; \quad \int_0^{\pi/3} \cos^4(\theta) \, d\theta = \frac{7\sqrt{3}}{64} + \frac{\pi}{8}.
\]

**Answer.**  
(a)

\[
\text{The intersection point satisfies } \cos \theta = 1/2 \text{ so that } \theta = \pm \pi/3, \ r = 1.
\]

\[
A = 2 \int_0^{\pi/3} \int_1^{2 \cos \theta} r \, dr \, d\theta = 4 \int_0^{\pi/3} \cos^2 \theta d\theta - \frac{\pi}{3} = \frac{\sqrt{3}}{2} + \frac{\pi}{3}
\]

---

1Here, we use the book notation for the x-moment of mass: \(M_y = \iint x \rho \, dA\) and for the y-moment of mass \(M_x = \iint y \rho \, dA\). Please note that in Kolokolnikov’s class notes, \(M_x\) and \(M_y\) was reversed.
(c) Solution 1: The y-moment of mass is zero (by symmetry). The x-moment is

\[
M_y = \int \int x \, dA \\
= 2 \int_0^{\pi/3} \int_1^{2 \cos \theta} r \cos \theta r \, dr \, d\theta \\
= 2 \int_0^{\pi/3} \cos \theta \left( \frac{r^3}{3} \right) _1^{2 \cos \theta} \, d\theta \\
= \frac{2}{3} \left\{ \int_0^{\pi/3} (8 \cos^4 \theta - \cos \theta) \, d\theta \right\} \\
= \frac{\sqrt{3}}{4} + \frac{2}{3} \pi.
\]

Then

\[
\bar{x} = \frac{M_y}{A} = \frac{\frac{\sqrt{3}}{4} + \frac{2}{3} \pi}{\frac{\sqrt{3}}{2} + \frac{3}{4}} \approx 1.321, \quad \bar{y} = 0. \tag{1}
\]

Solution 2:

Denote \( A_1 \) and \( A_2 \) the areas of the regions as shown, and denote \( M_{1y} \) and \( M_{2y} \) their x-moments of mass, and \( \bar{x}_1, \bar{x}_2 \) their centers of mass. We have:

\[
A_1 = \pi - A_2 = \frac{2}{3} \pi - \frac{\sqrt{3}}{2};
\]

\[
\bar{x}_1 = \frac{1}{2};
\]

\[
M_{1y} = \bar{x}_1 A_1 = \frac{1}{2} A_1;
\]

and

\[
M_{1y} + M_{2y} = \pi \cdot 1
\]

(the rhs is area of the circle \( A_1 + A_2 = \pi \) times its center of mass at 1). Then we obtain

\[
M_{2y} = \pi - A_1 \frac{1}{2}
\]

\[
= \frac{2}{3} \pi + \frac{\sqrt{3}}{4}.
\]

Then (1) follows.