1. Sketch the graphs of the following quadratic surfaces. For each of these, classify them (e.g. one-sheet hyperboloid, two-sheet hyperboloid, paraboloid, ellipsoid, saddle...).

(a) \( x^2 - y^2 + z^2 = 1 \)
(b) \( x^2 - y^2 + z^2 = -1 \)
(c) \( x^2 - y + z^2 = 1 \)
(d) \( x^2 - 2x - \frac{y^2}{4} + z^2 = 0 \)

**Answer.** *See hand-written solution*

2. The wind-chill index is modeled by the function

\[
W = 13 + 0.6T - 11v^{0.15} + 0.4Tv^{0.15}
\]

where \( T \) is the temperature and \( v \) is the wind speed. When \( T = 15^\circ C \) and \( v = 30 \text{ km/h} \), by how much would you expect the apparent temperature to drop if (a) the actual temperature decreases by 2\(^\circ\)C? (b) the wind speed increases by 2 km/h?

**Answer.** (a) \( W_T = 0.6 + 0.4v^{0.15} \) so that \( W_T(15, 30) = 1.266 \). The wind-chill index would decrease by 1.266 \( \times 2 = 2.53 \) degrees when the temperature decreases by 2 degree. 
(b) Similarly \( W_v = -1.65v^{-0.85} + 0.067v^{-0.85} \) and \( W_v(15, 30) = -0.14 \) so that the wind-chill temperature decreases by 0.14 \( \times 2 = 0.28 \) degrees when the wind increases by 2 km/h.

3. You are given a function with the property that \( f(x, y) = 2x + ayx^2 \) and \( f_y = x^3 + 1 \). Determine the constant \( a \). Hint: use equality of mixed partial derivatives.

**Answer.** Compute \( f_{xy} = ax^2 \) and compute \( f_{yx} = 3x^2 \). Equating the two yields \( a = 3 \).

4. Given a surface defined implicitly through

\[ x^2 - y^2 + z^2 + xy^3z = 2. \]

(a) Determine \( \frac{\partial z}{\partial x} \) at the point \((x, y, z) = (1, 1, 1)\).
(b) Determine \( \frac{\partial z}{\partial y} \) at the point \((x, y, z) = (1, 1, 1)\).
(c) Determine the equation of the plane tangent to this surface at the point \((x, y, z) = (1, 1, 1)\).

**Answer.** (a) Differentiate implicitly:

\[
2x + 2y\frac{\partial z}{\partial x} + y^3z + xy^3\frac{\partial z}{\partial x} = 0
\]

Plug in \( x, y, z = 1 \) to get

\[
2 + 2\frac{\partial z}{\partial x} + 1 + \frac{\partial z}{\partial x} = 0
\]

so that \( \frac{\partial z}{\partial x} = -1 \).

(b) Differentiate implicitly:

\[
-2y + 2z\frac{\partial z}{\partial y} + 3y^2xz + xy^3\frac{\partial z}{\partial y} = 0.
\]

Plug in \( x, y, z = 1 \) to get

\[
-2 + 2\frac{\partial z}{\partial y} + 3 + \frac{\partial z}{\partial y} = 0.
\]

so that \( \frac{\partial z}{\partial y} = -1/3 \).

(c) \( z - 1 = -1(x - 1) + (-1/3)(y - 1) \)
or

\[
z + x + y/3 = 7/3
\]
5. The wave heights \( h \) in the open sea depend on the speed \( v \) of the wind and the length of time \( t \) that the wind has been blowing at that speed. Values of the function \( h = h(v, t) \) are recorded in feet in the following table.

<table>
<thead>
<tr>
<th>Wind speed (knots)</th>
<th>( v )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
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<tbody>
<tr>
<td>10</td>
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<td>62</td>
<td>67</td>
<td>69</td>
<td></td>
</tr>
</tbody>
</table>

(a) On the \( t - v \) plane, make a rough sketch of several well-chosen contours of \( h(v, t) \). Indicate the values of contours chosen. Hint: you can print out the table above and draw the contours directly on the table.

(b) Estimate the values of \( h_v(30, 20) \) and \( h_t(30, 20) \). What is the practical interpretation of these values?

(c) Using part (c) and linear approximation, estimate \( h(33, 25) \) as best as you can.

**Answer.** (a)

\( h_v(30, 20) \approx \frac{17 - 8}{10} \approx \frac{28 - 17}{10} \approx \frac{28 - 8}{20} \approx 1 \)

This means the height grows by one unit per one knot increase in wind speed. Similarly,

\( h_t(30, 20) \approx \frac{17 - 16}{5} \approx \frac{18 - 17}{10} \approx \frac{18 - 16}{15} \approx 0.13 \)

So the wave height grows by (approximately) 0.13 units with every passing hour.

(d) Using linear approximation,

\[
h(33, 25) \approx h(30, 20) + 3h_v(30, 20) + 5h_t(30, 20) \\
= 17 + 3 \times 1 + 5 \times 0.13 = 20.7.
\]
6. Let \( u(x, t) \) be the temperature of a metal bar at a one-dimensional location \( x \) and time \( t \). Then \( u(x, t) \) satisfies the heat equation which has the following non-dimensional form:
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

(a) There is a solution of the Heat equation of form \( u(x, t) = e^{-\lambda t} \sin(mx) \). Determine a relationship between \( \lambda \) and \( m \).

(b) Find all solutions of the heat equation that have the form \( u(x, t) = F(t) \sin(mx) \) for some function \( F(t) \). Show that they are all some scalar multiple of the solution you found in part (a).

(c) Can you find any other solutions of the heat equation that are not of the form \( u(x, t) = F(t) \sin(mx) \)?

Answer. (a) \( \lambda = m^2 \).

(b) We get \( F'(t) = -m^2 F(t) \) whose solution is \( F(t) = Ae^{-m^2t} \) where \( A \) is arbitrary constant so that so the most general solution of this form is \( u(x, t) = Ae^{-m^2t} \sin(mx) \).

(c) Some other solutions not of this form include:
\[
\begin{align*}
  u(x, t) &= e^{-m^2t} \cos(mx); \\
  u(x, t) &= e^{-t} \sin(x) + e^{-4t} \sin(2x); \\
  u(x, t) &= Ax + B; \\
  u(x, t) &= \exp(m^2t) \exp(mx)
\end{align*}
\]

and there are many others...

7. The pressure, volume, and temperature of a mole of an ideal gas are related by the equation \( PV = 8.3T \), where \( P \) is measured in kilopascals, \( V \) in liters, and \( T \) in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 10 L to 12 L and the temperature decreases from 310 K to 305 K.

Answer. We have
\[
dP V + P \, dV = 8.3 \, dT
\]
with \( V = 10 \), \( dV = 2 \), \( T = 310 \), \( dt = -5 \), \( P = 8.3T/V = 257.3 \) which gives
\[
dP \times 10 + 257.3 \times 2 = 8.3 ( -5 )
\]
so that
\[
dP = -55.61.
\]
(a) \[ \frac{x^2}{r^2} + \frac{z^2}{r^2} - y^2 = 1 \]

Let \( r = \sqrt{x^2 + z^2} \), then the graph of \( r^2 - y^2 = 1 \) has the graph:

Here, \( r \) can be thought of as the \( xz \) plane and the desired surface is the above \( y \)-hyperbola revolved around the \( y \)-axis:

It's a one-sheet hyperboloid
(b) \[ x + \frac{2}{r^2} - y^2 = -1 \]

Sketch \[ r^2 - y^2 = -1 \]:

Then revolve about y-axis:

It's a two-sheet hyperboloid.
(c) \[ y = \sqrt[2]{\frac{x^2 + z^2}{r^2}} - 1 \Rightarrow y = \frac{r^2}{r^2} - 1 = 0 \]

So the surface is obtained by revolving the above parabola about the y-axis.

It's a paraboloid
(d) **Complete square:** \[ x^2 - 2x = (x-1)^2 - 1 \]

so we get

\[ (x-1)^2 - \frac{y^2}{4} + z^2 = 1 \]

**Change Var:** \( x-1 = \tilde{x}, \quad \frac{y}{2} = \tilde{y}, \quad z = \tilde{z} \)

then we get

\[ \tilde{x}^2 - \tilde{y}^2 + \tilde{z} = 1 \]

which is precisely the surface sketched in part (a):

So the surface \( y \text{(d)} \) is just above figure except that \( x = \tilde{x} + 1 \) (shift right) and \( y = 2\tilde{y} \) (stretch along y-axis).

Hyperboloid of one sheet.