1. 5. [Q57, in 12.3]: A molecule of methane is structured with the four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by the H–C–H combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about 109.5 degrees. Hint: Take the vertices of the tetrahedron to be the points \((1,0,0), (0,1,0), (0,0,1)\) and \((1,1,1)\), as shown in the figure. Then the centroid is \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\).

**Answer.** Form two vectors from the center to two of the vertices, for example \(u = (1,0,0) - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})\) and \(v = (0,1,0) - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\). Then \(u \cdot v = -\frac{1}{4} = |u||v|\cos \theta\) where \(\theta\) is the desired angle and \(|u| = |v| = \sqrt{3}/4\) so that \(\cos \theta = -1/3\) and \(\theta = \arccos(-1/3) \approx 109.471\).

2. Consider a triangle whose vertices are \(P = (1,0,0)\), \(Q = (0,1,0)\) and \(R = (0,0,2)\).

(a) Sketch the triangle \(PQR\) in three dimensional space.

(b) Find a vector that is in the direction perpendicular to the plane through \(PQR\).

(c) Find the area of triangle \(PQR\).

(d) Find the equation of the line that is perpendicular to the plane through \(PQR\) and goes through the point \((1,1,1)\).

**Answer.**

(a)

(b) Let \(u = Q - P = (-1,1,0)\), \(v = R - P = (-1,0,2)\). Then

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 1 & 0 \\
-1 & 0 & 2
\end{vmatrix} = 2\hat{i} + 2\hat{j} + \hat{k}
\]

which is the direction perpendicular to the plane.

(c) \(|u \times v|/2 = \sqrt{5}/2 = 3/2\).

(d) It's the line in the direction of \(u \times v\) going through the point \((1,1,1)\):

\((x,y,z) = (1,1,1) + (2,2,1)\ t.\)
3. Let $P$ be a point not on the plane that passes through the points $Q, R, S$.

(a) Show that the distance $d$ from to the point $P$ to the plane through $Q, R, S$ is

$$d = \frac{|a \cdot (b \times c)|}{|a \times b|}$$

where $a = QR$, $b = QS$, and $c = QP$.

**Answer.** The volume $|a \cdot (b \times c)|$ of the paralellopiped spanned by vectors $a, b, c$ equals to the distance $d$ from the point $P$ to the plane $QRS$, multiplied by the area $|a \times b|$ of the base: $|a \cdot (b \times c)| = d|a \times b|$, which is precisely the above formula

(b) Apply this formula in the case where $P = (0, 0, 0)$, $Q = (1, 0, 0)$, $R = (0, 1, 0)$, and $S = (0, 0, 2)$.

**Answer.** Here, $a = (-1, 1, 0)$, $b = (-1, 0, 2)$ and $c = (-1, 0, 0)$. Then

$$b \times c = -2j,$$

$$|a \cdot (b \times c)| = 2,$$

$$a \times b = 2i + 2j + \hat{k}$$

$$|a \times b| = \sqrt{5} = 3$$

$$d = \frac{2}{3}.$$  

4. Find the magnitude of the torque about $P$ if a 5-N force is applied as shown. Also find the direction of the torque vector.

**Answer.** The position vector is $r = (1, -1, 0)$ and has magnitude of $|r| = \sqrt{2}$. The force vector has magnitude of $|F| = 5$. The angle in between them is $\theta = 45 + 30 = 75$ degrees. So the magnitude of the torque is $|\tau| = 5 * \sqrt{2} * \sin 75^\circ = 6.83$. The right hand rule shows that $r \times F$ is oriented into the page (i.e. negative $\hat{k}$ direction). So $\tau = -6.83\hat{k}$.

5. The points $P = (0, 0, 0)$, $Q = (1, -1, 1)$, $R = (1, 0, 0)$ and $S = (1, 0, a)$ all lie on the same plane. Determine the value of $a$. Hint: use the formula for the volume of a parallelepiped.

**Answer.** Solution 1: Let $u = Q - P = (1, -1, 1)$, $v = R - P = (1, 0, 0)$, $w = S - P = (1, 0, a)$ Then

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = (0, 1, 1)$$

$$w \cdot (u \times v) = a$$

The three vectors are coplanar if and only if $w \cdot (u \times v) = 0$ so that $a = 0$.

Solution 2: If we choose $a = 0$ then $R = S$ and therefore all four points lie in the same plane.

6. (a) Find the equation of the line which is parallel to the planes $x + y + z = 1$ and $x = 2$, and which goes through the origin.

(b) Find the equation of the line which is the intersection of the planes in part (a).
Answer. (a) The normals to the two planes are \((1, 1, 1)\) and \((1, 0, 0)\). Their cross product is

\[
(1, 1, 1) \times (1, 0, 0) = (\hat{i} + \hat{k} + \hat{j}) \times \hat{i} = \hat{j} - \hat{k}
\]

So the desired line is in the direction \((0, 1, -1)\) going through the origin:

\[
(x, y, z) = (0, 0, 0) + t (0, 1, -1).
\]

(b) We just need any point which is in the intersection of the two planes, i.e. any solution to \(x + y + z = 1\) and \(x = 2\). For example \(x = 2, y = 0, z = -1\). So the desired line goes through \((2, 0, -1)\) and has the same direction as found in part (a):

\[
(x, y, z) = (2, 0, -1) + t (0, 1, -1).
\]