Wednesday October 19

Last time:

- The derivative as a function
  \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]
  when the limit exists...

- Rules on computing the derivative:
  
  Power rule: \( (x^n)' = nx^{n-1} \)
  Multiplication by constant: \( (cf(x))' = c \cdot f'(x) \)
  Sum rule: \( (f(x) + g(x))' = f'(x) + g'(x) \)
  \( (f(x) - g(x))' = f'(x) - g'(x) \).

Today:


Q. Find the equation of the line \( L \) which is tangent to the curve \( y = \frac{1}{x} \) at the point \((2,1)\).

\[ \text{We have:} \]
\[ (2,1) \text{ is a point on the line } L \]
\[ \text{We need the slope of the line.} \]

Slope is \( f'(1) \) where \( f(x) = \frac{1}{x} \).
We use the power rule to find
\[ f'(x) = (\frac{1}{x})' = (x^{-1})' = (-1)x^{-1-1} = -x^{-2} \]

We can now plug in \(x = 1\) and get
\[ f'(1) = -1^{-2} = -1 \] or slope of \(L\).

Thus, the equation of \(L\) is
\[ L: y - 1 = -1(x - 1) \]

Q. Find values for the coefficients \(a\), \(b\) and \(c\) so that the parabola \(f(x) = ax^2 + bx + c\) goes through the point \((0,1)\) and is tangent to the line \(y = 3x - 2\) at the point \((2,4)\).

Translate "words" into "mathy."

The parabola \(f(x) = ax^2 + bx + c\) goes through the point \((0,1)\).

\[ f(0) = 1 \]

Therefore,
\[ a \cdot 0^2 + b \cdot 0 + c = 1 \]

\[ c = 1 \]
The parabola \( f(x) = ax^2 + bx + c \) is tangent to the line \( y = 3x - 2 \) at the point \( (2, 4) \).

\[ f(2) = 4 \]

and

slope of tangent of \( f(x) \) at \( (2, 4) \) is:

\[ f'(2) = \text{slope of the line } y = 3x - 2 \]

Hence \( f'(2) = 3 \)

\[ f(2) = 4 \text{ gives } a \cdot 2^2 + b \cdot 2 + c = 4 \text{ or } 4a + 2b + c = 4. \]

We have found \( c = 1 \). Hence we get:

\[ 4a + 2b + 1 = 4 \]

\[ 4a + 2b = 3 \]

we got a relation between \( a \) and \( b \).

To actually find \( a \) and \( b \) we need one more relation. We are going to get it from:

\[ f'(2) = 3 \]

Let's see what this can give us.
What is $f'(x)$?

To compute it we need $f'(x)$.

$$f'(x) = (ax^2 + bx + c)'$$
$$= (ax^2)' + (bx)' + c'$$
$$= a \cdot (x^2)' + b \cdot (x)' + 0$$
$$= a \cdot 2x^{2-1} + b \cdot 1 \cdot x^{1-1}$$
$$= 2ax + b$$

For $x = 2$ we get

$$f'(2) = 4a + b.$$

(*) gives $f'(2) = 3$. This means

$$f'(2) = 4a + b = 3$$

We now have two equations for $a$ and $b$

1. $4a + 2b = 3$
2. $4a + b = 3$

To find $a$ and $b$ we need to solve this system.
(1) \[ 4a + 2b = 3 \]
(2) \[ 4a + b = 3 \]

Multiplying (2) by \(-2\) we get

\[
\begin{align*}
4a + 2b &= 3 \\
-8a - 2b &= -6
\end{align*}
\]

Adding these equations gives

\[ -4a = -3 \]

\[ \boxed{a = \frac{3}{4}} \]

Plugging \(a = \frac{3}{4}\) in (1) we get

\[
\frac{4}{4} \cdot \frac{3}{4} + 2b = 3
\]

or

\[ 3 + 2b = 3 \]

or

\[ 2b = 0 \]

\[ \boxed{b = 0} \]

We found \(a = \frac{3}{4}, b = 0, c = 1\)

Hence \(f(x) = \frac{3}{4} x^2 + 0 \cdot x + 1\)

\[ f(x) = \frac{3}{4} x^2 + 1 \]
1. Slope of the tangent line of the graph of \( f(x) = \frac{x^2 + 1}{x} \) at \((1, 3)\) is \( f'(1) \). We have to find \( f'(x) \).

\[
f(x) = \frac{x^2 + 1}{x} = x + x^{-1}
\]

\[
f'(x) = (x + x^{-1})' = (x)' + (x^{-1})' = 1 - x^{-2}
\]

For \( x = 1 \) we get

\[
f'(1) = 1 - 1^{-2} = 1 - 1 = 0
\]

\[
\boxed{f'(1) = 0}
\]

is the slope.

2. When does the derivative exist?

The derivative of \( f \) at \( x = a \) exists when the limit

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

exists and is a number.

We say that \( f \) is differentiable at \( x = a \)

If a function \( f(x) \) is differentiable at \( x = a \), then

\( f(x) \) is continuous at \( x = a \)
However, there are functions that are continuous at $x=a$ but not differentiable.

Examples:

(I) \[ f(x) = |x| \]

Graph:

The slope of the secant lines connecting $(0,0)$ and $(h,h)$ ($h>0$) is
\[ \frac{h-0}{h-0} = \frac{h}{h} = 1. \quad (1) \]

The slope of the secant lines connecting $(0,0)$ and $(-h,h)$ is
\[ \frac{h-0}{-h-0} = \frac{h}{-h} = -1. \quad (2) \]

Since the two slopes (1) and (2) are different, the two limits
\[ \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 1 \neq \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} \]
\[ = -1 \]
and the function is not differentiable at $x=0$, even though it is continuous.

Notice that $f(x)$ does not really have a tangent line at $x=0$ ...

\( f(x) = \sqrt{x} \)

Is $f(x)$ differentiable at $x=0$?

\( f'(x) = \frac{1}{2} \cdot x^{-1/2} = \frac{1}{2\sqrt{x}} \)

\( f'(0) = \frac{1}{0} \) not a number...

Looking at the graph,

we see that the tangent line of $f(x)$ at $(0,0)$ is the line $x=0$ (y axis) which has "infinite" slope.
Find the value of $a$ such that $f(x)$ below is differentiable at $x=0$.

$$f(x) = \begin{cases} x^2 + ax & , x \geq 0 \\ x & , x < 0 \end{cases}$$

Quick answer:
We compute

$$f'(x) = \begin{cases} 2x + a & , x \geq 0 \\ 1 & , x < 0 \end{cases}$$

For $f$ to be differentiable at $x=0$, we require $f'_{\text{top}}(0) = f'_{\text{bottom}}(0)$ to be continuous at $x=0$.

$$2 \cdot 0 + a = 1$$
$$a = 1$$

This answer is not really rigorous, but will get full points.

Rigorous answer:
We need to have that $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$ exists.

In other words, we need

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h}$$
We have

\[
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2 + ah - 0}{h} = \lim_{h \to 0^+} (h + a) = a
\]

\[
\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h}{h} = 1.
\]

Hence \[a = 1\].