Monday October 17

Review session: Monday October 24 in IBLC 182, 6-8 pm.
Last time: Derivative of a function at a point.

Key facts:

- \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \) \( (\text{Number!}) \)

*derivative of \( f \) at \( a \)

read: \( f \) prime of \( a \)

- \( f'(a) \) = instantaneous rate of change of \( f \) at \( x = a \).
- \( f'(a) \) is also:
  
  slope of the tangent line of the graph of \( f(x) \)
  
  at \( (a, f(a)) \).

\[
\text{equation of tangent line at } (a, f(a)) \]

\[
y - f(a) = f'(a)(x - a)
\]

(Recall: equation of line with slope \( m \) passing through \( (x_1, y_1) \) is

\[
y - y_1 = m \cdot (x - x_1)
\]
Remark: The number $f'(a)$ tells us things about the graph of $f(x)$.

If $f'(a) > 0 \Rightarrow$ tangent line at $(a, f(a)) = \Delta$ graph has positive slope \( \Rightarrow \) graph is increasing \( \Rightarrow m_t > 0 \) \( \Rightarrow \) graph is increasing.

\[ \text{Example:} \]

\[ f(x) = x^2 \]

\[ f'(2) > 0 \text{ since graph is increasing close to } x = 2 \]
(tangent line looks like \( \)).

\[ f'(-2) < 0 \text{ since graph is decreasing close to } x = -2 \]
(tangent line looks like \( \)\).
What about $x=0$?

$\Rightarrow f'(0) = 0$.

The tangent of $f(x) = x^2$ at $(0,0)$ is the line $y = 0$, which has slope 0. Hence $f'(0) = 0$.

Recall: last time we computed $f'(a) = 2a$ for any $a$. ($f(x) = x^2$).

Today:
The derivative as a function and "easy" ways to compute it.

We can think of the derivative as a function

$\begin{align*}
\text{input} & \quad \rightarrow \quad f'(x) \\
\text{output} & \quad \rightarrow \quad f'(t)
\end{align*}$

input is time $t$, output is 'instantaneous velocity at time $t$.

or...

$\begin{align*}
\text{input} & \quad \rightarrow \quad f'(x) \\
\text{output} & \quad \rightarrow \quad \text{slope of the tangent line of the graph of $f$ at $(x, f(x))$.}
\end{align*}$
Examples

* \( f(x) = x^2 \)
  As a function \( f'(x) = 2x \)
  For \( x = 1 \) \( f'(1) = 2 \cdot 1 = 2 \)
  For \( x = 2 \) \( f'(2) = 2 \cdot 2 = 4 \)
  For \( x = -2 \) \( f'(-2) = 2 \cdot (-2) = -4 \)
  and so on...

* \( f(x) = x^3 \)
  \( f'(x) = 3x^2 \)

* \( f(x) = x^4 \)
  \( f'(x) = 4x^3 \)

In general...

Power rule

If \( f(x) = x^\nu \)

\[ f'(x) = \nu \cdot x^{\nu-1} \]

for any number \( \nu \).

e.g. \( f(x) = x^5 \); \( f'(x) = 5x^4 \)

\( f(x) = \sqrt{x} = x^{1/2} \); \( f'(x) = \frac{1}{2} \cdot x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}} \)
\[ f(x) = \frac{3}{x^{5/3}} = x^{-5/3} \]

\[ f'(x) = \frac{5}{3} x^{5/3 - 1} = \frac{5}{3} x^{2/3} \]

\[ f(x) = \frac{1}{x} = x^{-1} \]

\[ f'(x) = -x^{-2} \]

If we wanted to compute this with the definition of the derivative, we would have lots of work to do.

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \]

\[ = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \left( \frac{(x+h)x - x(x+h)}{(x+h)x} \right) \]

\[ = \lim_{h \to 0} \frac{x - (x+h)}{(x+h)x} \]

\[ = \lim_{h \to 0} \frac{-h}{(x+h)x} \]

\[ = \lim_{h \to 0} \frac{-h}{(x+h)x} \left( \frac{-1}{h} \right) \]

\[ = \lim_{h \to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2} = -x^{-2} \]
Multiplication by a number:

\[(c \cdot f(x))' = c \cdot f'(x)\] for any number \(c\).

\[\begin{align*}
(2 \cdot x^4)' &= 2 \cdot (x^4)' = 2 \cdot 4x^3 = 8x^3 \\
(\sqrt{x-2})' &= \sqrt{x-2} = \sqrt{2} \cdot (-2) \cdot (x-2)^{-1} = -2\sqrt{2} \cdot x^{-3} = -2\sqrt{2} \cdot x^{-3}
\end{align*}\]

Derivative of constant functions:

\[f(x) = c, \quad f'(x) = (c)' = 0\]

[If you don't move then your velocity is 0.]

Formulas:

\[\begin{align*}
(f(x) + g(x))' &= f'(x) + g'(x) \\
(f(x) - g(x))' &= f'(x) - g'(x)
\end{align*}\]

Example: \(f(x) = x^{1/2} + 2x^3 + 5\)

\[\begin{align*}
 f'(x) &= (x^{1/2} + 2x^3 + 5)' \\
 &= (x^{1/2})' + (2x^3)' + (5)' \\
 &= \frac{1}{2} x^{-1/2} + 2(3x^2)' + 0 \\
 &= \frac{1}{2} x^{-1/2} + 6x^2 \\
 &= \frac{1}{2} x^{-1/2} + 6x^2
\end{align*}\]