Monday October 3

Last time: Computing limits from the formula of a function $f(x)$

- If $f(x)$ exists then
  - $\lim_{x \to a} f(x) = f(a)$

- If $f$ is not defined at $x = a$
  - $\lim_{x \to a} f(x) = \frac{0}{0}

- If $\frac{f(a)}{f(x)} = \frac{1}{0}$
  - Then the limit is $+\infty$ if $f(x) > 0$ close to $a$
  - $-\infty$ if $f(x) < 0$ close to $a$.

Continuity

A function $f(x)$ is continuous if you can draw its graph without lifting your pencil.

- continuous

Not continuous.

Here I lifted the pencil.
A function \( f(x) \) is continuous at \( x = a \) if close to \( x = a \), I can draw the graph of \( f(x) \) without lifting the pencil.

In other words, the function "doesn't jump" at \( x = a \).

\[
\begin{align*}
\text{Continuous at } x = 2 \\
\text{not} \\
\text{Continuous at } x = -1
\end{align*}
\]

Mathematically,

A function \( f(x) \) is continuous at \( x = a \) if

- \( f(a) \) is defined, \( f \) is defined at \( a \).

- \( \lim_{x \to a} f(x) \) exists

- \( \lim_{x \to a} f(x) = f(a) \).
Question: Can a function be defined at $a$, if \( \lim_{x \to a} f(x) \) exists but \( \lim_{x \to a} f(x) \neq f(a) \)?

\[ \Rightarrow \text{YES} \]

Here when $a = 1$ we have

\[ f(1) = 2 \quad (\text{exists}) \]

But \( \lim_{x \to 1} f(x) = 1 \neq f(1) = 2 \).

This function is \textbf{NOT} continuous at $x = 1$.

\[ \text{Continuous functions - examples} \]

\[ f(x) = ax^2 + bx + c \]

\[ f(x) = \frac{1}{x^2} \]
linear functions: \( f(x) = mx + b \)

Moreover, products and quotients of continuous functions are also continuous (at the points they are defined)

That's why on Friday we could plug in \( a \) in \( f(x) \) when \( f(a) \) existed and compute

\[
\lim_{x \to a} f(x) = f(a)
\]

Our functions were continuous.
• Let \( f(x) = \begin{cases} x^2, & x \geq 0 \\ x, & x < 0 \end{cases} \)

Is \( f(x) \) continuous?

⇒ We have to check that \( f(x) \) is continuous at every \( x = a \).

Two ways:

1. From the graph

   Graph:

   We can draw this graph without lifting the pencil.

   Hence, \( f(x) \) is a continuous function.

2. From the formula:

   \( f(x) \) has a different formula when \( x \geq 0 \) (\( f(x) = x^2 \)) and when \( x < 0 \) (\( f(x) = x \)).

   We take three cases:

   - \( x = a > 0 \)
   - \( x = a < 0 \)
   - \( x = 0 \)
When \( x = a > 0 \) we want to check if \( f(x) \) is continuous at \( a > 0 \).
To do so, notice that we can find a "small" interval close to \( a \) where all \( x > 0 \) and \( f(x) = x^2 \).
We know that quadratics are continuous functions.
Hence, \( f(x) \) is continuous at \( x = a > 0 \).

When \( x = a < 0 \) we will see that \( f(x) \) is continuous at \( a < 0 \).
Notice that we can find a small interval close to \( a < 0 \) where \( x < 0 \)
and \( f(x) = x \).
We know that linear functions are continuous.
Hence \( f(x) \) is continuous when \( a < 0 \).

When \( x = 0 \).
We want to see that \( f(x) \) is continuous at \( x = 0 \).

We have to verify 3 conditions:

(i) \( f(0) \) is defined.
(ii) \( \lim_{{x \to 0^-}} f(x) = \lim_{{x \to 0^+}} f(x) = f(0) \).
(iii) \( \lim_{{x \to 0}} f(x) = f(0) \).
For (i) we compute \( f(0) = 0^2 = 0 \) \( \square \)

For (ii) \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0 \) \( \square \)

plugging it in:

Here \( x < 0 \), this is why I chose the formula \( f(x) = x \).

\( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0^2 = 0 \) \( \square \)

Here \( x > 0 \), hence \( f(x) = x^2 \).

Therefore, \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \) \( = \lim_{x \to 0} f(x) \)

\( = f(0) = 0 \)

Condition (iii) is also verified \( \square \).

Thus, \( f(x) \) is continuous at \( x = 0 \).