Friday October 7,

- Midterm review session (choose a time...)

An important property of continuous functions.

**Intermediate Value Theorem (I.V.T.)**

**Assumptions:**
- \( f \) is a continuous function on \([a, b]\).
- \( f(a) < l < f(b) \) or \( f(b) < l < f(a) \) for some \( l \) in \( \mathbb{R} \).

**Conclusion:**
- There is a number \( c \) such that \( f(c) = l \).
Examples:

If \( f(x) \) is a continuous function on \([-1, 3]\):

1. \( f(-1) = -2 \) and \( f(3) = 2 \), prove that:
   1. There is a number \( c \) such that \( f(c) = 0 \).
   2. There is a number \( m \) such that \( f(m) = 1 \).

1. We know that:
   - \( f \) is a continuous function on \([-1, 3]\).
   - \( f(-1) = -2 < 0 < f(3) = 2 \).

Therefore, using the I.V.T. we conclude that there is some \( c \) such that \( f(c) = 0.5 \).

2. We know that:
   - \( f \) is a continuous function on \([-1, 3]\).
   - \( f(-1) = -2 < 1 < f(3) = 2 \).

Therefore, using the I.V.T. we conclude that there is some number \( m \) such that \( f(m) = 1 \).
Graph explanation/idea:

We know that the graph of $f$ is a continuous curve connecting $(-1, f(-1))$ and $(3, f(3))$.

\[ (-1, -2) \quad (3, 2) \]

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Part 1. says that the graph should cross the axis. There is some $c$ such that $f(c) = 0$.

Part 2. says that there is some $m$ (on the x-axis) such that $(m, 1)$ is on the graph of $f$.

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Q. Consider the function $f(x) = x^3 - x + 1$.

Prove that there is a number $-2 < c < -1$ such that $f(c) = 0$.

No. $f(x)$ is continuous on $[-2, -1]$.

\[
\begin{align*}
    f(-2) &= (-2)^3 - (-2) + 1 = -8 + 2 + 1 = -8 + 3 = -5 \\
    f(-1) &= (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1
\end{align*}
\]

Hence
\[ f(-2) < 0 < f(-1) = 1. \]

Therefore, by the I.V.T. we know that there is some \( c, -2 < c < -1 \) such that \( f(c) = 0 \).

Q. Prove that the function \( f(x) = x^3 - 15x + 1 \) has two roots in the interval \([-4, 2]\).

**Step 1:** Compute values of \( f \) at the endpoints of the interval.

\[
\begin{align*}
  f(-4) &= (-4)^3 - 15(-4) + 1 \\
         &= -64 + 60 + 1 \\
         &= -3. \\

  f(2) &= 2^3 - 15 \cdot 2 + 1 \\
        &= 8 - 30 + 1 \\
        &= -21.
\end{align*}
\]

**Step 2:** Need two numbers \( x_1, x_2 \) such that \( f(x_1) < 0 < f(x_2) = 0 \). I will apply the I.V.T. in two different intervals.

I have \( f(-4) < 0 \) \( f(2) < 0 \).

I need a positive \( f \) value \( y \). Let's try \( x = 0 \) ...

\[ f(0) = 1 > 0 \]

Now \( f(-4) < 0 < f(0) = 1 \)? \( f \) is continuous on \([-4, 0]\). Therefore, by the I.V.T. there exists \( x \) with \(-4 < x, x < 0\).
Such that $f(x_1) = 0$.

Moreover, $f(0) = 1 > 0 > f(9) = -2$

$f$ is continuous on $[0, 9]$
therefore there is $x_2$ with $0 < x_2 < 9$

such that $f(x_2) = 0$

We found $-4 < x_1 < 0$ and $0 < x_2 < 2$

with $f(x_1) = f(x_2) = 0$.

We are done.

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