Friday September 23

Last time... (Section 1.0 of the book)

Suppose we throw an icecube from a restaurant located 41 meters above the ground.

It's height at time $t$ (distance from the ground) ($t$ seconds) is given by $h(t) = -t^2 + 4$.

If the icecube reaches the ground with speed less than $-3$ m/sec then it breaks.

Will our icecube break?

$\rightarrow$ We have to compute the velocity of the icecube at the instant $t = 2$ sec. (why $t = 2$ sec? 

That's when the icecube reaches the ground.

This is hard. 

Instead: Approximate the instantaneous velocity of the icecube at $t = 2$ sec.

Compute the average velocity in a small time interval including $t = 2$ sec.

One small interval like this is $[1.9, 2]$. It has length $2 - 1.9 = 0.1$. 

Small means with small length. Recall: length of interval $[a, b]$ is $|a - b| = b - a$.
Average velocity is \[ \frac{h_{\text{final}} - h_{\text{in}}}{t_{\text{final}} - t_{\text{in}}} \]

\[ h(t) = -t^2 + 4 \]

Plug in \( t = 2 \) ...

\[ \frac{h(2) - h(1.9)}{2 - 1.9} \]

Important:

\[ -2^2 + 4 - \left( - (1.9)^2 + 4 \right) \]

\[ \frac{0.1}{0.1} \]

\[ (1.9)^2 - 4 \]

\[ = -3.9 \text{ m/sec.} \]

\[ \text{The average velocity of the ice cube is almost} \]

\[-3.9 \text{ m/sec. This is less than } -3 \text{ m/sec.}\]

Hence, the ice cube will almost certainly break.

• Can we approximate the instantaneous velocity at \( t = 2 \text{ sec.} \) better?

Yes! Make the interval smaller.

Complete the table:

<table>
<thead>
<tr>
<th>Time interval</th>
<th>Average velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>[1.9, 2]</td>
<td>-3.9</td>
</tr>
<tr>
<td>[1.92, 2]</td>
<td>-3.92</td>
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<tr>
<td>[1.93, 2]</td>
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<tr>
<td>[1.97, 2]</td>
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</tr>
<tr>
<td>Time interval</td>
<td>Average velocity m/sec</td>
</tr>
<tr>
<td>--------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>[1.98, 2]</td>
<td>-3.98</td>
</tr>
</tbody>
</table>

Looks like the instantaneous velocity at $t = 2$ sec is going to be $-4$ m/sec.

The computations took some time.....
Can I do them all at once?

Sure!

Average velocity in $[t, 2] = \frac{h(2) - h(t)}{2 - t}$

$h(t) = -t^2 + 4$

$-2^2 + 4 - (-t^2 + 4) = \frac{-2 + t^2}{2 - t}$

Do not compute the square $(2^2)$ move leave it for the end.

$= \frac{-2 + t^2}{2 - t}$
\[
\frac{t^2 - 2^2}{2 - t} = \frac{(t-2)(t+2)}{2-t} = \frac{(t-2)(t+2)}{-(t-2)}
\]

Recall: \[a^2 - b^2 = (a-b)(a+b)\]

\[= - (t+2) \quad \star\]

We now know the average velocity in the time interval \([t, 2]\).

Without doing calculations as before, we can plug in numbers into the formula \(\star\) and get e.g.

Average velocity in \([1.999, 2]\)

is \[-(1.999 + 2) = -3.999 \text{ m/sec}.\]

This is even closer to \(-4 \text{ m/sec}\).

The instantaneous velocity at \(t = 2\) sec is actually \(-4 \text{ m/sec}\).
Geometric interpretation

The average velocity in \([t, 2]\) = slope of the secant line joining the points \((2, h(2))\) and \((t, h(t))\)

In graph

If we go closer and closer to 2, the secant lines are approaching the tangent line at \(t = 2\).
Let \( f(t) = 2t^2 + t \). Approximate the slope of the tangent line at the graph of this function at \( t=2 \).

The problem is the same as before.

Think of the analogy:

- \( f(t) \) \( \leftrightarrow \) height at time \( t \)
- slope of tangent \( \leftrightarrow \) instantaneous velocity
- slope of secant line \( \leftrightarrow \) average velocity in time interval \([t, 2]\).

We will compute the slope of the secant line joining \((t, f(t))\) and \((2, f(2))\) (think \([t, 2]\) small time interval).

This is \( \frac{f(2) - f(t, 4)}{2 - 1.9} = \)
\[
\frac{2 \cdot 2^2 + 1 - (2 \cdot (1.9)^2 + 1)}{0.1} = 7.8
\]

↑ a bit messy...

It's easier to find the slope of the secant line joining \((t, f(t))\) and \((2, f(2))\) and plug in \(t = 1.9\) afterwards.

Slope is \(\frac{f(2) - f(t)}{2 - t}\)

\[
2 \cdot 2^2 + 1 - (2 \cdot t^2 + 1) = 2 - t
\]

\[
2 \cdot 2^2 + 1 - 2 \cdot t^2 - 1 = 2 - t
\]

\[
2 \cdot (2^2 - t^2) = 2 - t
\]

\[
2 \cdot (2-t)(2+t) = 2 - t
\]

\[
2 \cdot (2+t); \text{ when } t = 1.9 \text{ I get } \sqrt{2 \cdot (3.9)} = 7.8
\]
When \( t \) gets close to 2, the velocity gets close to \( 2 \cdot (2 + 2) = 8 \). Slope of the tangent line at \( t = 2 \).