Problem 1. (6 points.) Let the following three data points be given

\[(x_0, y_0) = (-1, 0), \quad (x_1, y_1) = \left(\frac{1}{2}, 2\right), \quad (x_2, y_2) = (2, -1).\]

a) Assume that you want to find an interpolation function \(f(x)\) that has the form

\[
f(x) = \begin{cases} 
  p_1(x) = a_1 x^2 + b_1 x + c_1, & -1 \leq x \leq \frac{1}{2} \\
  p_2(x) = a_2 x^2 + b_2 x + c_2, & \frac{1}{2} \leq x \leq 2 
\end{cases}
\]

(i) Write down the equations that \(a_j, b_j\) and \(c_j\), with \(j \in \{1, 2\}\), must satisfy such that \(f(x)\) is continuous and fits the given data points.

\[
\begin{align*}
p_1(-1) &= 0 : \quad a_1 - b_1 + c_1 = 0 \\
p_1\left(\frac{1}{2}\right) &= 2 : \quad \frac{1}{4} a_1 + \frac{1}{2} b_1 + c_1 = 2 \\
p_2\left(\frac{1}{2}\right) &= 2 : \quad \frac{1}{4} a_2 + \frac{1}{2} b_2 + c_2 = 2 \\
p_2(2) &= -1 : \quad 4 a_2 + 2 b_2 + c_2 = -1
\end{align*}
\]

(ii) Assume in addition that \(f'(x_0) = 0\) and that \(f'(x_2) = 3\). Write the corresponding equations as well as the equations from part (i) into one matrix equation for the unknowns \([a_1, b_1, c_1, a_2, b_2, c_2]^T\).

\[
f'(x_0) = 0 \iff p'_1(x_0) = 0 \iff p'_1(-1) = 0 : -2a_1 + b_1 = 0 \quad (1P)
\]

\[
f'(x_2) = 3 \iff p'_2(x_2) = 3 \iff p'_2(2) = 3 : 4a_2 + b_2 = 3
\]

\[
\begin{bmatrix}
1 & -1 & 1 & 0 & 0 & 0 \\
-\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{2} & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 2 & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
b_1 \\
c_1 \\
a_2 \\
b_2 \\
c_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
2 \\
-1 \\
3
\end{bmatrix}
\]
b) Assume now that you want to find just one polynomial of degree 2, i.e.

\[ p(x) = ax^2 + bx + c, \]

that fits all the given data points.
Write down the linear system that needs to be solved to find the unknowns \( a, b, c \).

\[
\begin{bmatrix}
1 & -1 & 1 \\
1/4 & 1/2 & 1 \\
4 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix} = 
\begin{bmatrix}
0 \\
2 \\
-1
\end{bmatrix}
\]  
(2p)

**Problem 2.** (4 points.) Consider the boundary value problem

\[
\begin{align*}
f''(x) - (\cos(x) + 1)f(x) &= \frac{2}{\pi} \\
f(0) &= -1, \quad f(2\pi) = 0
\end{align*}
\]

Write down the finite difference matrix equation for the corresponding finite difference approximation with \( N = 4 \).

The finite difference approximation is given by

\[
(L + (\Delta x)^2 Q) F = \begin{bmatrix}
A \\
B
\end{bmatrix}
\quad \text{with} \quad F = \begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

and

\[
L = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
0 & -2 & 1 & 0 \\
0 & 0 & -2 & 1
\end{bmatrix}, \quad \Delta x = \frac{2\pi - 0}{4} = \frac{\pi}{2}
\]

\[(1p)\]

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \text{since}
\]

\[
q_1 = q(x_1) = -(\cos(\frac{\pi}{2}) + 1) = -1
\]

\[
q_2 = q(x_2) = -(\cos(\pi) + 1) = 0
\]

\[
q_3 = q(x_3) = -(\cos(\frac{3\pi}{2}) + 1) = -1
\]

\[
(\Delta x)^2 R = \begin{bmatrix}
\frac{\pi}{2} & 0 & 0 & 0 \\
0 & \frac{\pi}{2} & 0 & 0 \\
0 & 0 & \frac{\pi}{2} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus

\[
(L + \frac{\pi^2}{4} Q) F = \begin{bmatrix}
f_0 \\
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

\[(1p)\]