Last time: Spline interpolation

1. Cubic splines: \( p_j(x) = A_j x^3 + B_j x^2 + C_j x + D_j \) for \( 1 \leq j \leq n \)

   Have \( 4(n-1) \) unknowns \( A_j, B_j, C_j, D_j \), \( 1 \leq j \leq n \).

   Need: \( 4(n-1) \) equations to solve for the \( 4(n-1) \) unknowns

2. By imposing the three conditions for splines plus a fourth condition for the behaviour of the interpolation at the endpoints, we found \( 4(n-1) \) equations:

   Condition C1: \( p_j(x_j) = y_j \), \( p_j(x_{j+1}) = y_{j+1} \) \( \quad \) [2(n-1) equations]

   Condition C2: \( p_j'(x_j) = p_j'(x_{j+1}) \) \( \quad \) [(n-2) equations]

   Condition C3: \( p_j''(x_j) = p_j''(x_{j+1}) \) \( \quad \) [(n-2) equations]

   Condition C4: \( p_j'''(x_i) = 0 \), \( p_{n-1}'''(x_n) = 0 \) \( \quad \) [2 equations]

3. We saw an explicit example for \( n=3 \) datapoints \( x \) and derived the corresponding 8 equations by hand. Then we solved the resulting linear system with the help of Matlab.

4. We started with an efficient and numerically stable method to solve cubic splines.
Let's impose the conditions (C1) - (C4) to the polynomials
\[ p_j(x) = A_j(x) y_j + B_j(x) y_{j+1} + C_j(x) z_j + D_j(x) z_{j+1} \]
with unknowns \( z_j, \ldots, z_{n-1} \) and functions \( A_j(x), B_j(x), C_j(x), D_j(x) \) defined in the last lecture.

(C1)
\[ p_j(x_j) = y_j \quad \text{and} \quad p_j(x_{j+1}) = y_{j+1} \]
\[ p_j(x_j) = A_j(x_j) y_j + B_j(x_j) y_{j+1} + C_j(x_j) z_j + D_j(x_j) z_{j+1} \]
\[ = \begin{bmatrix} A_j(x_j) & B_j(x_j) & C_j(x_j) & D_j(x_j) \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} \]
\[ = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} \]
\[ = y_j \]

Thus \( p_j(x_j) = y_j \) is automatically fulfilled!
\[ p_j(x_{j+1}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = y_{j+1} \]

Thus also \[ p_j(x_{j+1}) = y_{j+1} \] is automatically satisfied!

\[ P_j''(x_{j+1}) = P_{j+1}'(x_{j+1}) \]

\[ P_j''(x_{j+1}) = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_j \\ y_{j+1} \\ z_j \\ z_{j+1} \end{bmatrix} = z_{j+1} \]

\[ P_{j+1}'(x_{j+1}) = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{j+1} \\ y_{j+2} \\ z_{j+1} \\ z_{j+2} \end{bmatrix} = z_{j+1} \]

\[ \Rightarrow P_j''(x_{j+1}) = z_{j+1} = z_{j+1} = P_{j+1}'(x_{j+1}) \]

So also this condition is automatically satisfied!
\( p''_1(x_1) = 0 \)

\[
p''_1(x_1) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \tau_{12} \end{bmatrix} = z_1
\]

\[ \Rightarrow z_1 = 0 \quad \Rightarrow \text{this is a new equation} \]

\( p''_{n-1}(x_n) = 0 \)

\[
p''_{n-1}(x_n) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \\ \tau_{n-1} \end{bmatrix} = z_n
\]

\[ \Rightarrow z_n = 0 \quad \Rightarrow \text{this is a new equation} \]
\[ p'_{j+1}(x'_{j+1}) = p'_{j+1}(x'_{j+1}) \quad (\star) \]

Explicit calculations give

\[
(\star) \iff \frac{(x_{j+1} - x_j)}{6} z_j + \frac{(x_{j+2} - x_j)}{3} z_{j+1} + \frac{(x_{j+2} - x_{j+1})}{6} z_{j+2}
\]

\[
= \frac{y_{j+2} - y_{j+1}}{x_{j+2} - x_{j+1}} - \frac{y_{j+1} - y_j}{x_{j+1} - x_j}
\]

Since we have this for \( j = 1, \ldots, n-2 \) we get \( n-2 \) additional equations for \( z_1, \ldots, z_n \)

\[ \therefore \text{Therefore we have found} \quad 2 + n - 2 = n \text{ equations!} \]
Using matrix notation

\[ \begin{bmatrix}
1 \\
\frac{x_2 - x_1}{6} \\
\frac{x_3 - x_2}{3} \\
\frac{x_4 - x_3}{6} \\
0 \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 \\
z_5 \\
z_6 \\
z_7 \\
\end{bmatrix} = \begin{bmatrix}
y_3 - y_2 \\
y_4 - y_3 \\
y_5 - y_4 \\
y_6 - y_5 \\
y_7 - y_6 \\
y_8 - y_7 \\
y_9 - y_8 \\
\end{bmatrix}
\]

\[ z = S^{-1} b \]

Need to solve \( S z = b \) => \( z = S^{-1} b \)

Calculating determinants and inverses of tri-diagonal matrices can be done very efficiently.
Finite differences approximation

Goal: Find approximate numerical solution of a certain differential equation (DE)

We will focus on

\[ f''(x) + q(x) f(x) = r(x) \]

where \( q(x), r(x) \) are known functions and \( f(x) \) is unknown

Example: \( q(x) = 0 \), \( r(x) = 4 \) \( \Rightarrow \) \( f''(x) = 4 \) \( \square \)

We solve \( \square \) exactly:

\[ f'(x) = 4x + C_1 \]

\[ f(x) = 2x^2 + C_1 x + C_2 \]

general solution of \( \square \)
How to get a unique solution?

As always we need more conditions

Possibility 1: Specify initial condition of $f$ and $f'$

\[
\begin{align*}
  f''(x) &= 4 \\
  f(0) &= 1 \quad f'(0) = 2
\end{align*}
\]

Initial value problem (IVP)

Back to the example: general solution $f(x) = 2x^2 + C_1 x + C_2$

Impose:

\[
\begin{align*}
  f(0) = 1 &\quad \Rightarrow \quad C_2 = 1 \\
  f(0) = 2 &\quad \Rightarrow \quad C_1 = 2
\end{align*}
\]

Thus $f(x) = 2x^2 + 2x + 1$ solve the IVP above.
Possibility 2: Specify the "boundary values" of \( f'(x) \) for boundary value problem (BVP)

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= 4 \\
f(0) &= 1, \quad f(1) = 0
\end{align*}
\]

Again back to the example

\[
\begin{align*}
f(0) &= 1 \quad \Rightarrow \quad c_2 = 1 \\
f(1) &= 0 \quad \Rightarrow \quad 2 + c_1 + 1 = 0 \quad \Rightarrow \quad c_1 = -3
\end{align*}
\]

\[f(x) = 2x^2 - 3x + 1\]

Problem: Even simple-looking DE's are hard to solve OR they do not have any exact solution at all.

Example: \( \frac{d^2}{dx^2} f(x) + \cos(x) f(x) = x^2 \) has no explicit solution