Last time: Interpolation (Fitting a function $f(x)$ to given data points)

Task: Find a function $f(x)$ such that $f(x_1) = y_1, f(x_2) = y_2, \ldots, f(x_n) = y_n$ where $(x_i, y_i), i = 1, \ldots, n$ are given data points.

Problem: This is not a well-posed. These exist infinitely many functions $f$ that interpolate the finitely many given data points!

Solution: Restrict the set of allowed functions

Lagrange Interpolation

Given: $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$ where the $x_i$ are all distinct

Want: A polynomial $p(x)$ of degree $n-1$ that fits the data

$$p(x) = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

$n$ unknowns!

In order to fit the data the polynomial $p(x)$ needs to satisfy

$$p(x_1) = y_1, p(x_2) = y_2, \ldots, p(x_n) = y_n$$

$n$ equations

Moreover the $n$ equations (or better to say the corresponding matrix equation) have a very specific form $\mapsto$ Vandermonde matrix
Last time continued: Vandermonde matrix

Fact: i) \( \det(V) = (-1)^{\frac{n(n-1)}{2}} \prod_{i>j} (x_i - x_j) \)

ii) \( \det(V) \neq 0 \) as long as \( x_i \neq x_j \) for all \( i \neq j \).

Back to Lagrange Interpolation:
Want: Coefficients \( a_1, a_2, \ldots, a_n \) for the polynomial \( p(x) \).
Have: Matrix equation

\[
\begin{pmatrix}
    x_1^{n-1} & x_1^{n-2} & \cdots & x_1 \\
    x_2^{n-1} & x_2^{n-2} & \cdots & x_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n^{n-1} & x_n^{n-2} & \cdots & x_n \\
\end{pmatrix}
\begin{pmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n \\
\end{pmatrix}
= 
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n \\
\end{pmatrix}
\]

Solution: Since the \( x_i, 1 \leq i \leq n \), are all distinct the Vandermonde matrix generated by the \( x_i \) is invertible. Therefore

\[
\begin{pmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_n \\
\end{pmatrix}
= \left( \begin{pmatrix}
    x_1^{n-1} & x_1^{n-2} & \cdots & x_1 \\
    x_2^{n-1} & x_2^{n-2} & \cdots & x_2 \\
    \vdots & \vdots & \ddots & \vdots \\
    x_n^{n-1} & x_n^{n-2} & \cdots & x_n \\
\end{pmatrix} \right)^{-1}
\begin{pmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_n \\
\end{pmatrix}
\]
and we have found the polynomial \( p(x) \).
Example: Data points $(0, 1), (2, 1), (1, 3)$

We want to find $p(x) = a_1 x^2 + a_2 x + a_3$

We have the equations

$p(0) = 1 \implies a_1(0)^2 + a_2(0) + a_3 = 1$
$p(2) = 1 \implies a_1(2)^2 + a_2(2) + a_3 = 1$
$p(1) = 3 \implies a_1(1)^2 + a_2(1) + a_3 = 3$

In matrix notation

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{det}(V) = (-1)^{3(2)} \prod_{i > j} (x_i - x_j) = -\frac{3(2)}{2} \frac{(x_3-x_2)(x_3-x_1)(x_2-x_1)}{-1 \cdot 1 \cdot 2} = 2$$
Thus we know $V$ is invertible and we calculate the inverse

\[ V^{-1} = \begin{bmatrix} 0.5 & 0.5 & -1 \\ -1.5 & -0.5 & 2 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ V \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \iff V^{-1} V \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = V^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

\[ \iff \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = V^{-1} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \]

\[ \begin{bmatrix} 0.5 & 0.5 & -1 \\ -1.5 & -0.5 & 2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} \iff p(x) = -2x^2 + 4x + 1 \]
Piecewise linear polynomial interpolation

\[ p_3(x) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} x + c_i \]

for some constant \( c_i \).

Then small changes in \( y \) will affect the interpolation only locally and by a small amount.

To do this the interpolation function \( f(x) \) needs to have some properties.
Properties of a piecewise polynomial interpolation $f(x)$

1. $f(x)$ is continuous and $f(x_i) = y_i$ for $1 \leq i \leq n$

2. $f'(x)$ exists and is continuous

3. For each $x$ except for $x_1, x_2, \ldots, x_n$ all higher derivatives $f''(x), f'''(x), \ldots, f^{(n)}(x)$ exist and have right/left limits as $x \to x_j$ for $1 \leq j \leq n$

(Note: we can have jumps at the $x_j$)

Definition: Splines are "piecewise polynomials" satisfying the conditions 1-3 above.

$$f(x) = \begin{cases} 
  p_1(x) & x_1 \leq x < x_2 \\
  p_2(x) & x_2 \leq x < x_3 \\
  \vdots \\
  p_{n-1}(x) & x_{n-1} \leq x \leq n 
\end{cases}$$
Q: What polynomials $p_j$ should we use?

Answer: It turns out that cubic splines (all $p_j$ of degree three) have several good properties including "variation minimization."

The linear equation for cubic splines

$$p_j(x) = A_j x^3 + B_j x^2 + C_j x + D_j$$

for $j = 1, \ldots, n-1$

Unknowns: $A_j, B_j, C_j, D_j$ for $1 \leq j \leq n-1$

$\Rightarrow 4(n-1)$ unknowns
To find a (hopefully) unique solution we need 4 \((n-1)\) equations. To obtain those we use the properties of the spline

\(\text{a) Imose continuity condition } \bigcirc\)

\[
\begin{align*}
& p_j(x_j) = y_j \\
& p_j(x_{j+1}) = y_{j+1} \\
\end{align*}
\]

\(\Rightarrow \text{ fits the data to the polynomials and gives us } 2(n-1) \text{ equations}\)

\(\text{b) Condition } \bigcirc\)

\[
\begin{align*}
& p'_j(x_{j+1}) = p'_{j+1}(x_{j+1}) \\
\end{align*}
\]

\(\Rightarrow \text{ this gives us } n-2 \text{ equations}\)
c) Condition (3) (Note: second derivatives are enough here)

\[ p_j''(x_{j+1}) = p_{j+1}''(x_{j+1}) \quad \text{for} \quad j = 1, \ldots, n-2 \]

\[ \Rightarrow \quad n-2 \text{ equations} \]