MATH 307: Applied Linear Algebra

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Office hour: Monday 11:30 am - 1 pm LSK 300 B

Lecture I

Which of the following is linear?

1. $y = 2x$ ✓
2. $y = 2x^2 + 3$ ✗
3. $y - 5x = -3$ ✓
4. $x + y + 2xy = 5$ ✗ non-linear.
A system with two linear equations and two unknowns

\[
\begin{align*}
I & \quad a_{11} x_1 + a_{12} x_2 = b_1 \\
II & \quad a_{21} x_1 + a_{22} x_2 = b_2
\end{align*}
\]

where \( x_1, x_2 \) are unknowns, the rest are given constants

Simple case: \( a_{12} = a_{21} = 0 \)

\[\Rightarrow \] Reduces to the 1-d case.

Row operations preserve the solution and can be used to find the solution (if it exists)

\[
\begin{align*}
 &a_{11} \cdot II \\
 &\begin{align*}
 a_{11} x_1 + a_{12} x_2 = b_1 \\
 a_{11} a_{21} x_1 + a_{11} a_{22} x_2 &= a_{11} b_2
\end{align*}
\end{align*}
\]

\[
\begin{align*}
 &II - a_{21} \cdot I \\
 &\begin{align*}
 a_{11} x_1 + a_{12} x_2 = b_1 \\
 0 + a_{11} a_{22} x_2 - a_{12} a_{21} x_2 &= a_{11} b_2 - a_{21} b_1
\end{align*}
\end{align*}
\]
So we see: 
\[(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - a_{21}b_1\]

If \[a_{11}a_{22} - a_{12}a_{21} \neq 0\], we have a unique solution.

But what if \[a_{11}a_{22} - a_{12}a_{21} = 0\]?

Then we have 2 cases:

i) \[a_{11}b_2 - a_{21}b_1 = 0\], then \(0x_2 = 0\), and thus we have \(\infty\)-many solutions.

ii) \[a_{11}b_2 - a_{21}b_1 \neq 0\], then \(0x_2 = b \neq 0\), thus no solution.

Now, let's rewrite this system using matrix notation:

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

In this (since it's 2x2) we can check if \( A \) is invertible by the determinant: \( \det A = a_{11}a_{22} - a_{12}a_{21} \)
\[ \det A \neq 0 \Rightarrow \text{There is an inverse of } A, \text{ called } A^{-1} \]

Property: \[ A^{-1}A = I_2 = AA^{-1} \]

\[ Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \]

\[ \Rightarrow I_2x = A^{-1}b \Rightarrow x = A^{-1}b \]

We see that we have a unique solution when \( \det(A) \neq 0 \).

**Remarks:**

Using the inverse can be generalized to any system of \( n \) equations and \( n \) unknowns when \( \det A \neq 0 \).

What about \( m \times n \) matrices with \( m \times n \)?

In that case a simple form of \( m \times n \) matrix similar to the identity matrix for square matrices exists.
Example:

\[ R = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \]

2x3 matrix, 3x1 vector

Want to solve \( RX = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \)

\[
\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

is in REF (reduced row echelon form)

\[
\begin{cases}
1x_1 + 0x_2 + 5x_3 = 1 \\
0x_1 + 1x_2 + 1x_3 = 0
\end{cases}
\]

(as a linear system)

\[
\begin{align*}
x_1 &= 1 - 5x_3 \\
x_2 &= -x_3 \\
x_3 \text{ is free variable}
\end{align*}
\]

Thus for all real numbers, there is a solution given by setting \( x_3 = t \).
Check by plugging in:

\[ R\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) = R\left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + R\left( x_3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \]

\[ = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Remark: Any arbitrary linear system with \( m \) equations and \( n \) unknowns can be written in the form \( Ax = b \).

Important fact: Any matrix can be reduced to its ref by Gaussian elimination.
Example:
\[
\begin{align*}
2x_1 + 4x_2 + 6x_3 &= 2 \\
x_1 + x_2 + 4x_3 &= 1
\end{align*}
\Leftrightarrow
\begin{equation*}
A\mathbf{x} = \mathbf{b}
\end{equation*}
\]
where
\[
A = \begin{bmatrix}
2 & 4 & 6 \\
1 & 1 & 4
\end{bmatrix}
\quad b = \begin{bmatrix}
2 \\
1
\end{bmatrix}
\quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

To solve this system of linear equations consider the augmented matrix
\[
\begin{bmatrix}
A & \mid & b
\end{bmatrix} = \begin{bmatrix}
2 & 4 & 6 & | & 2 \\
1 & 1 & 4 & | & 1
\end{bmatrix}
\]

\begin{align*}
\text{I} \leftrightarrow \text{II} & \implies \begin{bmatrix}
1 & 1 & 4 & | & 1 \\
2 & 4 & 6 & | & 2
\end{bmatrix} \\
\text{II} / 2 & \implies \begin{bmatrix}
1 & 1 & 4 & | & 1 \\
1 & 2 & 3 & | & 1
\end{bmatrix} \\
\text{II} - \text{I} & \implies \begin{bmatrix}
1 & 1 & 4 & | & 1 \\
0 & 1 & -1 & | & 0
\end{bmatrix}
\end{align*}
Thus to find the solutions of $Ax=b$ we need to find the solutions of
\[
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
0
\end{bmatrix}
\]

Strategy to solve the general form of a system of linear equations

i) Apply gaussian elimination to the augmented matrix to get into row reduced echelon form (rref)

ii) Solve the rref-system by identifying pivots and free variables.
Norms of vectors & matrices

1) Norms of vectors

A norm $\|v\|$ of a vector is a generalization of the length of a vector or the size of a vector.

Example: $\mathbb{R}^2$:

$\|\begin{bmatrix} 1 \\ 2 \end{bmatrix}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$

This is the so-called euclidean norm, or 2-norm or $L_2$-norm.

Definition (Norm)

A norm $\|v\|$ of a vector is a real-valued function satisfying the following three conditions:

1. (Non-negativity) $\|v\| \geq 0$ for all $v$.
2. (Positive definiteness) $\|v\| = 0$ if and only if $v = 0$.
3. (Triangle inequality) $\|v + w\| \leq \|v\| + \|w\|$ for all $v$ and $w$. 

Graphical representation:

- Axis labels: $x_1$, $x_2$.
- Vector $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$.
- Vector $v$ is plotted on the plane.
\( \| x \| \geq 0 \) with \( \| x \| = 0 \iff x = 0 \)

ii) \( \| s x \| = |s| \| x \| \) for any scalar \( s \) and vector \( x \)

iii) For all vectors \( x, y \), we have
\[ \| x + y \| \leq \| x \| + \| y \| \] (triangle inequality)

**Example:** (Norm on \( \mathbb{C}^n \))

Let \( x \in \mathbb{C}^n \)

1) Euclidean norm (\( \ell_2 \)-norm) on \( \mathbb{C}^n \)
\[ \| x \|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2} \]

\[ \| [2i] \|_2 = \sqrt{|2i|^2 + 1^2} = \sqrt{5} \]