The University of British Columbia
18 March 2015

Common Midterm for All Sections of MATH 105 (Version 1)

Closed book examination

Time: 60 minutes

Last Name  Zho
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Signature

Student Number

MATH 105  Section Number:

Special Instructions:
No memory aids are allowed. No calculators. No communication devices. Show all your work; little or no credit will be given for a numerical answer without the correct accompanying work. If you need more space than the space provided, use the back of the previous page.

Rules governing examinations

- Each candidate must be prepared to produce, upon request, a UBCcard for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
  (a) Having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners.
  (b) Speaking or communicating with other candidates.
  (c) Purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
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- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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[5] 1. Find the derivative of the function

\[ f(x) = \int_{1}^{1+x^2} \frac{\cos \sqrt{t-1}}{\pi} \, dt \]

at the point \( x = \frac{\pi}{3} \):

\[ f'(x) = \frac{\cos \sqrt{1+x^2-1}}{\pi} \cdot 2x \]

\[ = \frac{2x \cos \sqrt{x^2}}{\pi} \]

\[ f'(\frac{\pi}{3}) = \frac{2 \frac{\pi}{3} \cos \frac{\pi}{3}}{\pi} \]

\[ = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\pi} \]

\[ = \frac{1}{3} \]

(a) [4] Evaluate \( \int \frac{x^2}{\sqrt{1-5x^3}} \, dx \).

Let \( u = 1-5x^3 \), then \( du = -15x^2 \, dx \).

\[
\begin{align*}
\int \frac{x^2}{\sqrt{1-5x^3}} \, dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{-15} \\
&= -\frac{1}{15} \int u^{-\frac{1}{2}} \, du \\
&= -\frac{1}{15} \cdot \frac{1}{\frac{1}{2}} \cdot u^{\frac{1}{2}} + C \\
&= -\frac{2}{15} u^{\frac{1}{2}} + C \\
&= -\frac{2}{15} \sqrt{1-5x^3} + C
\end{align*}
\]

(b) [5] Evaluate \( \int_1^e x^2 \ln x \, dx \).

Let \( u = \ln x \), \( dv = x^2 \, dx \) \( \Rightarrow v = \frac{x^3}{3} \), \( du = \frac{1}{x} \, dx \).

So \( \int x^2 \ln x \, dx = \int u \, dv \)

\[
= uv - \int v \, du \\
= \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx \\
= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} \, dx \\
= \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx \\
= \frac{x^3}{3} \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + C \\
= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C
\]

So \( \int_1^e x^2 \ln x \, dx = \left[ \frac{x^3}{3} \ln x - \frac{1}{9} x^3 \right]_1^e \\
= e^3 + \frac{1}{3} - \frac{1}{9} - \frac{1}{3} \cdot 1
\]

(a) [5] Evaluate \( \int_0^1 \frac{x^2}{(\sqrt{4 - x^2})^3} \, dx \).

Let \( x = 2 \sin(\theta) \), then \( dx = 2 \cos(\theta) \, d\theta \).

\[
4 - x^2 = 4 \cos^2(\theta).
\]

\[
\int_0^1 \frac{x^2}{(\sqrt{4 - x^2})^3} \, dx = \int \frac{4 \sin^2(\theta)}{(2 \cos^3(\theta))^3} \cdot 2 \cos(\theta) \, d\theta
\]

\[
= \int \frac{8 \sin^2(\theta) \cdot \cos(\theta)}{8 \cos^3(\theta)} \, d\theta
\]

\[
= \int \tan^2(\theta) \, d\theta
\]

\[
= \int (\sec^2(\theta) - 1) \, d\theta \quad \text{since} \quad \tan^2(\theta) + 1 = \sec^2(\theta)
\]

\[
\frac{x}{\sqrt{4 - x^2}} + C
\]

\[
\sin(\theta) = \frac{x}{2}, \text{ then } \theta = \sin^{-1}
\]

(b) [5] Evaluate \( \int \frac{5x - 1}{2x^2 + x - 1} \, dx \).

Since \( 2x^2 + x - 1 = (x+1)(2x-1) \)

Then \( \frac{5x - 1}{2x^2 + x - 1} = \frac{A}{x+1} + \frac{B}{2x-1} \)

\[
\frac{5x - 1}{2x^2 + x - 1} = \frac{A}{x+1} + \frac{B}{2x-1}
\]

\[
5 \cdot x - 1 = A(2x - 1) + B(x + 1)
\]

\[
= (2A + B) x - A + B
\]

So \( 2A + B = 5 \)

\[-A + B = 1 \Rightarrow A = 2, B = 1
\]

So \( \frac{5x - 1}{2x^2 + x - 1} = \frac{2}{x+1} + \frac{1}{2x-1} \)

\[
\Rightarrow \int \frac{5x - 1}{2x^2 + x - 1} \, dx = 2 \int \frac{dx}{x+1} + \int \frac{1}{2x-1} \, dx
\]

\[
= 2 \ln|x+1| + \frac{1}{2} \ln|\frac{1}{2x-1}| + C
\]

\[
= 2 \ln|x+1| + \frac{1}{2} \ln|\frac{1}{2x-1}| + C
\]
4. Evaluate the following integrals.

(a) \[ \int_{e}^{\infty} \frac{dx}{x \ln^{99} x} \cdot \]

In fact, \( \int_{e}^{b} \frac{dx}{x \ln^{99} x} = \lim_{a \to 1^+} \int_{a}^{b} \frac{dx}{x \ln^{99} x} \)

For \( \int \frac{dx}{x \ln^{99} x} \), we have

\[ \int \frac{dx}{x \ln^{99} x} = \int \frac{du}{u^{98}} \quad \text{let } u = \ln^{99} x \quad \text{then } du = \frac{dx}{x} \]

\[ = \int \frac{1}{98} \cdot u^{-98} + C \]

\[ = \int \frac{1}{98} \cdot \frac{1}{\ln^{99} x} + C \]

So \( \int_{e}^{b} \frac{dx}{x \ln^{99} x} = \lim_{a \to 1^+} \left[ -\frac{1}{98} \cdot \ln^{99} x \right]^{b}_{a} \]

(b) \[ \int_{1}^{2} \frac{dx}{(x-1)^{4}} \cdot \]

In fact, \( \int_{1}^{2} \frac{dx}{(x-1)^{4}} = \lim_{a \to 1^+} \int_{a}^{2} \frac{dx}{(x-1)^{4}} \)

For \( \int \frac{dx}{(x-1)^{4}} \), then

\[ \int \frac{dx}{(x-1)^{4}} = \int \frac{du}{u^{3}} \quad \text{let } u = x - 1 \quad \text{then } du = dx \]

\[ = -\frac{1}{3} u^{-3} + C \]

\[ = -\frac{1}{3} \cdot \frac{1}{(x-1)^{3}} + C \]

So \( \int_{1}^{2} \frac{dx}{(x-1)^{4}} = \lim_{a \to 1^+} \left[ -\frac{1}{3} \cdot \frac{1}{(x-1)^{3}} \right]^{2}_{a} \)

\[ = \lim_{a \to 1^+} \left[ \frac{1}{3} \cdot \frac{1}{(2-1)^{3}} - \frac{1}{3} \cdot \frac{1}{(a-1)^{3}} \right] \]

\[ = \lim_{a \to 1^+} \left[ \frac{1}{3} \cdot \frac{1}{(a-1)^{3}} - \frac{1}{3} \right] \]
[7] 5. A function \( y = f(x) \) has the following properties:

\[
\frac{dy}{dx} = e^{2x} e^{-y}, \quad y(0) = \ln 5.
\]

Find the function \( f(x) \).

\[
\text{Since} \quad \frac{dy}{dx} = e^{2x} e^{-y}
\]

\[
\text{Then} \quad e^y \frac{dy}{dx} = e^{2x}
\]

\[
\int e^y \, dy = e^{2x} \, dx
\]

\[
\int e^y \, dy = \int e^{2x} \, dx
\]

\[
\text{Then} \quad e^y = \frac{1}{2} e^{2x} + C
\]

Since \( y(0) = \ln 5 \), then

\[
5 = e^{\ln 5} = \frac{1}{2} + C
\]

\[
\Rightarrow C = 5 - \frac{1}{2} = \frac{9}{2}
\]

\[
\text{So} \quad e^y = \frac{1}{2} e^{2x} + \frac{9}{2}
\]

\[
\Rightarrow y = \ln \left( \frac{1}{2} e^{2x} + \frac{9}{2} \right).
\]
[4] 6. Evaluate \[ \int \sin^4 x \sec^2 x \, dx. \]

In fact, we have

\[ \int \sin^4 x \sec^2 x \, dx = \int \sin^4 x \cdot \frac{1}{\cos^2 x} \, dx \]

\[ = \int \frac{\sin^4 x}{\cos^2 x} \, dx \]

\[ = \int \left[ 1 - \cos^2 x \right]^2 \, dx \quad \text{since} \quad \sin^2 x + \cos^2 x = 1 \]

\[ = \int \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^2 x} \, dx \]

\[ = \int [\sec^2 x - 2 + \cos^2 x] \, dx \]

\[ = \tan x - 2x + \int \frac{1 + \cos 2x}{2} \, dx \quad \text{since} \quad \cos 2x = 2\cos^2 x - 1 \]

\[ = \tan x - 2x + \frac{1}{2}x + \frac{1}{4} \sin 2x + C \]

\[ = \tan x - \frac{3}{2}x + \frac{1}{4} \sin 2x + C \]
[8] Let \( f(x) \) be a continuous function defined on \([9, 25]\). It is known that \( +1 \leq f'(x) \leq 2 \) and \( 0 \leq f''(x) \leq 1 \) for all \( x \) in \([9, 25]\). Estimate the absolute error in the Trapezoidal Rule approximation for \( \int_{3}^{5} f(x^2) \, dx \) with \( n = 8 \) subintervals.

You may take it for granted that the absolute error in the Trapezoidal Rule approximation for \( \int_{a}^{b} f(x) \, dx \) with \( n \) subintervals is bounded by \( \frac{k(b - a)^3}{12n^2} \), where \( k \) is a real number such that \( |f''(x)| \leq k \) for all \( x \) in \([a, b]\).

By the assumption, we have

1. \( |f'(x)| \leq 2 \) for all \( x \in [9, 25] \)
2. \( |f''(x)| \leq 1 \) for all \( x \in [9, 25] \)

Let \( g(x) = f(x^2) \) for all \( x \in [3, 17] \). Then,

\[
\begin{align*}
g'(x) &= f'(x^2) \cdot 2x = 2xf'(x^2) \\
g''(x) &= 2x \cdot f''(x^2) + 2xf'(x^2) \\
&= 2f'(x^2) + 4x^2f''(x^2)
\end{align*}
\]

Then

\[
|g'(x)| = 2|x f'(x^2)| \leq 2 \cdot 1.5 \cdot 2 \quad \text{by 1} \\
\leq 20 \quad \text{for all } x \in [3, 17]
\]

\[
|g''(x)| = |4x^2f'(x^2) + 2f'(x^2)| \\
\leq 4x^2|f'(x^2)| + 2|f'(x^2)| \\
\leq 4 \cdot 2.2 + 2 \cdot 2 \quad \text{by 1 and 2} \\
= 104
\]

Hence, by Theorem, we have

\[
\int_{3}^{5} f(x^2) \, dx \leq 104 \cdot \frac{(5-3)^2}{12 \cdot 8^2} = \frac{104 \cdot 4}{12 \cdot 64} = \frac{13}{24}
\]