Note: Your work will be graded both on correctness of the mathematics, and the clarity of presentation. Prove your statements, or justify your calculations, as appropriate.

1. Prove that for all $|z| < 1$, the following infinite product converges to the given limit

$$
(1 + z)(1 + z^2)(1 + z^4)(1 + z^8)(1 + z^{16}) \cdots = \frac{1}{1 - z}.
$$

Soln: By induction, we show the partial product satisfies

$$
\prod_{k=0}^{n} (1 + z^{2^k}) = 1 + z + z^2 + z^3 + \cdots + z^{(2^n + 1) - 1},
$$

which is the initial part of a geometric series with limit $1/(1 - z)$. Note the convergence holds for $|z| < 1$, but it is not uniform.

It is also possible to use induction to show directly that

$$
(1 - z) \prod_{k=0}^{n} (1 + z^{2^k}) = 1 - z^{2^{n+1}}.
$$

The limit $L$ of the partial products satisfies $(1 - z)L = 1$, also giving the result we desire.

2. Find an infinite series expansion for $\cos \pi z$. You may use our construction of $\sin \pi z$ as a useful model. Prove your result.

Soln: It’s funny, nobody commented that the question said series expansion, when what I wanted was the product expansion. Anyhow, everyone gave me a product expansion. So all is well!
From Hadamard’s theorem, and noting that \( \cos(\pi z) \) has order of growth one, with zeros at \( n + 1/2 \) for all integers \( n \), we know

\[
\cos(\pi z) = e^{Az+B} \prod_{n \in \mathbb{Z}} E_1(1 - \frac{z}{n + 1/2}).
\]

Putting pairs in positive/negative pairs, we see the paired canonical factors simplify (because their exponential parts cancel) which gives

\[
E_1(1 - \frac{z}{n + 1/2})E_1(1 + \frac{z}{n + 1/2}) = (1 - \frac{z^2}{(n + 1/2)^2}) = (1 - \frac{4z^2}{(2n + 1)^2}).
\]

Thus

\[
\cos(\pi z) = e^{Az+B} \prod_{n=0}^{\infty} (1 - \frac{4z^2}{(2n + 1)^2}).
\]

Since \( \cos \) is an even function, as is the product (it only involves \( z^2 \)), we conclude that \( e^{Az+B} \) is even. This only happens if \( A = 0 \). Evaluating both sides of the equation at \( z = 0 \) gives

\[
\cos(0) = 1 = e^B \prod_{n} 1,
\]

so we know \( e^B = 1 \). Thus we have

\[
\cos(\pi z) = \prod_{n=0}^{\infty} (1 - \frac{4z^2}{(2n + 1)^2}).
\]

3. Prove that \( e^z = z \) has infinitely many solutions. (Hint, try using Hadamard’s theorem.)

Soln: Suppose there are only finitely many roots. Let \( P(z) \) be a polynomial with the same zeros as \( e^z - z \), including multiplicities. The ratio \( (e^z - z)/P(z) \) has only removable singularities, so it is entire and non-zero. Thus it can be expressed as an exponential. By Hadamard, we know

\[
e^z - z = e^{Az+B}P(z).
\]

Differentiate twice to get

\[
e^z = e^{Az+B}(P''(z) + 2AP'(z) + A^2P(z)).
\]

The left hand side \( e^z \) has no zeros, so the polynomial \( P''(z) + 2AP'(z) + A^2P(z) \) must be a constant. This happens only if \( A = 0 \) or if \( P(z) \) is a constant.
In the case $A = 0$, our initial equation is $e^z - z = e^B P(z)$. Differentiating enough times kills the polynomial side, so we have $e^z \equiv 0$, which is impossible.

In the case $P(z)$ constant, we can absorb the constant into $e^B$ and our initial equation becomes $e^z - z = e^{Az+B}$. Differentiating twice shows $A = 1$.

Thus we have $e^z - z = e^z e^B$ and thus $z = e^z (1 - e^B)$. Which is impossible as $z$ has exactly one zero in the complex plane, while $e^z (1 - e^B)$ has either no zeros, or is identically zero.

From these contradictions, we conclude $e^z - z$ has infinitely many zeros.

4. a) Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, and has at least 2 fixed points. Prove that $f(z)$ is necessarily the function $f(z) = z$.

b) Find a function $f : \mathbb{D} \to \mathbb{D}$ which is a holomorphic bijection, and has no fixed points.

Soln a): Suppose one fixed point is called "a". Compose $f$ with the corresponding FLT $\phi_a(z) = (z - a)/(az - 1)$, then the map $g = \phi_a \circ f \circ \phi_a$ is also a bijection of the unit disk, with two fixed points, one of which is $0$. By Schwars’ lemma, the function $g$ must be a rotation (since $|g(b)| = |b|$ at the other fixed point). But the rotation must be the null rotation since $g(b) = b$; so $g(z) = z$. Hence $f(z) = \phi_a \circ \phi_a)(z) = z$.

Soln b): The only bijections of the disk are the FLTs of the form a rotation times a $\phi_a$ as above. Easy to check that $\phi_a$ has a fixed point, so we need to try something else. Try this:

$$g(z) = -\phi_a(z) = -\frac{z - a}{az - 1}.$$  

Then $g(z) = z$ iff $-(z - a) = a(az - 1)$, or equivalently $a = \pi z^2$. For non-zero $a$, this says the only fixed points are $z = \pm a/|a|$, which are on the unit circle and not in the unit disk.

5. a) Prove that the real-valued function $u(x, y)$ defined by

$$u(x, y) = \text{Re} \left( \frac{i + z}{i - z} \right) \text{ and } u(0, 1) = 0$$

is harmonic on the unit disk and that vanishes on the boundary.

b) Verify that $u(0, 0) = 1$, which is obviously larger than the values of $u$ on the boundary (which are all zero).

c) Explain why the observation in part b) does not contradict the maximum modulus principle for harmonic functions.
Soln a): \( u \) is the real part of the holomorphic function \( f(z) = (i + z)/(i - z) \) on the disk, hence by theorem, \( u \) is harmonic on the disk. On the boundary, we have \( z \bar{z} = 1 \), so

\[
\frac{i + z}{i - z} = \frac{(i + z)(-i - \bar{z})}{(i - z)(-i - \bar{z})} = \frac{1 - i(z + \bar{z}) - z\bar{z}}{1 + z\bar{z}} = \frac{1 - i2\text{Real}(z) - 1}{2} = -i \cdot \text{Real}(z),
\]

which is pure imaginary. So its real part is zero, and so \( u \) is zero on the boundary.

Soln b): Come on. Just plug in.

Soln c): the reason this is not a contraction is that \( u \) is not continuous on the closed disk. Indeed, \( u(0, y) = \text{Real}(i + iy)/(i - iy) = (1 + y)/(1 - y) \) which tends to infinity at \( y \to 1^- \).

6. a) Show that \( f(z) = -(z + 1/z)/2 \) defines a conformal map from the half disk \( \{|z| < 1, \text{Im}(z) > 0\} \) onto the upper half plane.

b) Include a sketch of the map, showing where some interesting points map as well. Use your judgement as to what is “interesting.”

Soln a): The function has a pole at \( z = 0 \) and the derivative is zero at \( z = \pm 1 \). All these points are outside that half disk, so we will get a conformal map so long as we can show the map is 1-1 on the disk. To see it is 1-1, we solve:

\[
-(z + 1/z)/2 = w,
\]

which is equivalent to solving the quadratic

\[
z^2 + 2wz + 1 = 0.
\]

This has roots \( w \pm \sqrt{w^2 - 1} \). Since the product of these root is 1, that means one has magnitude \( \geq 1 \), and the other \( \leq 1 \). Thus at most one of them is inside the unit disk, so the map is 1-to-1 on the disk.

On the boundary of the half-disk, the function maps the half-circle \( z = e^{i\theta} \) to the line \([-1, 1]\), the interval \([-1, 0]\) goes to \([1, \infty)\) and the interval \((0, 1]\) to \((-\infty, -1]\). This surrounds the upper half plane (with the right orientation), so by the principle of the argument, the map \( f \) takes the open half disk onto the upper half plane. Thus \( f \) is 1-to-1, onto, and holomorphic from the upper half disk to the upper half plane.

Soln b): I was hoping people would use some software to make an interesting plot of curves.

It is worth noting that polar coordinates in the disk are worth considering, so if we write \( z = re^{i\theta} \), then

\[
f(re^{i\theta}) = \frac{-1}{2}(re^{i\theta} + \frac{1}{r}e^{-i\theta}) = x + iy
\]

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\]
gives
\[ x = -\frac{1}{2}(r + \frac{1}{r}) \cos(\theta), \quad y = -\frac{1}{2}(r - \frac{1}{r}) \sin(\theta). \]

For fixed \( r \), this describes an ellipse with foci at \( \pm 1 \). It is in the upper half plane for \( 0 < \theta < \pi \), and \( 0 < r < 1 \) since the resulting \( y \) value is positive. For fixed \( \theta \), it describes a hyperbola. As \( r \) gets close to zero, the ellipse blows out to infinity. As \( r \) gets close to 1, the ellipse gets close to the line \([-1, 1]\).

Figure 1: Domain (top) and range (bottom). Red circles map to ellipses, black radial lines to hyperbolas.