Assignment 4 - Complex Analysis

MATH 440/508 – M.P. Lamoureux

Due Friday, Nov 27 at lecture

Note: Your work will be graded both on correctness of the mathematics, and
the clarity of presentation. Prove your statements, or justify your calcula-
tions, as appropriate.

1. Prove that for all \(|z| < 1\), the following infinite product converges to the
given limit

\[
(1 + z)(1 + z^2)(1 + z^4)(1 + z^8)(1 + z^{16}) \cdots = \frac{1}{1 - z}.
\]

2. Find an infinite series expansion for \(\cos \pi z\). You may use our construction
of \(\sin \pi z\) as a useful model. Prove your result.

3. Prove that \(e^z = z\) has infinitely many solutions. (Hint, try using
Hadamard’s theorem.)

4. a) Suppose \(f : \mathbb{D} \to \mathbb{D}\) is holomorphic, and has at least 2 fixed points.
Prove that \(f(z)\) is necessarily the function \(f(z) = z\).

   b) Find a function \(f : \mathbb{D} \to \mathbb{D}\) which is a holomorphic bijection, and has
no fixed points.

5. a) Prove that the real-valued function \(u(x, y)\) defined by

\[
u(x, y) = \text{Re} \left( \frac{i + z}{i - z} \right) \text{ and } u(0, 1) = 0
\]

is harmonic on the unit disk and that vanishes on the boundary.

   b) Verify that \(u(0, 0) = 1\), which is obviously larger than the values of \(u\)
on the boundary (which are all zero).

   c) Explain why the observation in part b) does not contradict the max-
imum modulus principle for harmonic functions.
6. a) Show that \( f(z) = -(z + 1/z)/2 \) defines a conformal map from the half disk \( \{|z| < 1, \text{Im}(z) > 0\} \) onto the upper half plane.

b) Include a sketch of the map, showing where some interesting points map as well. Use your judgement as to what is “interesting.”