Assignment 2 - Complex Analysis

MATH 440/508 – M.P. Lamoureux

Due Friday, Oct 16 at lecture

Note: Your work will be graded both on correctness of the mathematics, and the clarity of presentation. Prove your statements, or justify your calculations, as appropriate.

1. Suppose \( \Gamma \) is a simple closed curve in the complex plane, piecewise smooth, oriented in the counterclockwise direction. What is the value of the integral
   \[ \int_{\Gamma} z \, dz. \]
   Prove your result.
   (Hint: you did this for circles and squares in Assignment 1. The answer is a simple measure of some geometric property of the curve.)

2. a) Does there exist a holomorphic function \( f \) on the unit disk with
   \[ f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{n^2}, \text{ for } n = 1, 2, 3, \ldots \]
   Is it unique?
b) Does there exist a holomorphic function \( f \) on the unit disk with
   \[ f\left(\frac{1}{n}\right) = f\left(\frac{-1}{n}\right) = \frac{1}{n^3}, \text{ for } n = 1, 2, 3, \ldots \]
   Is it unique?
   Prove your statements.

3. Use the Cauchy integral formula theorem to compute
   a) \( \int_{|z|=1} z^n \sin(z) \, dz \)  \hspace{1cm} b) \( \int_{|z|=2} z^n(1-z)^m \, dz \)  \hspace{1cm} c) \( \int_{|z|=\rho} |z-z_0|^{-4} |dz| \) \quad |z_0| \neq \rho.
   Note: \( m, n \) are integers, positive or negative. \( |dz| = \sqrt{x'(t)^2 + y'(t)^2} \, dt \).
4. a) Use the residue theorem to compute
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^4} \, dx. \]
b) For any even integer \( 2N \), how would you compute
\[ \int_{-\infty}^{\infty} \frac{1}{1+x^{2N}} \, dx. \]
This formula simplifies to a single evaluation of cosecant, times something like \( N \). With that hint, can you compute this integral in general? Show your work.

5. Prove, that if \( f \) is a non-constant, entire function satisfying
\[ |f(z)| < |z|^N \]
for some fixed \( N \) and all \( z \) sufficiently large, then \( f \) has at most \( N \) zeros in the complex plane.
(This suggests some weird connection between growth and zeros!)

6. a) Show that the real part \( u(x, y) = \text{Real} \{ f(x + iy) \} \) of an analytic function \( f \) in the open unit disk \( D \) is harmonic. That is, \( u \) is twice continuously differentiable (in the real variables sense) and satisfies the PDE
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \]
b) Show that given a (real-valued) harmonic function \( u(x, y) \) on the open unit disk, there exists another harmonic function \( v(x, y) \) such that the function
\[ f(z) = f(x + iy) = u(x, y) + iv(x, y) \]
is analytic in the unit disk. (Hint: Cauchy-Riemann equations)
c) Assume \( u(x, y) \) is harmonic in the open unit disk, and continuous on its closure. Show we can compute its values inside the disk by the real integral
\[ u(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1-r^2}{1-2r\cos(\theta - \phi) + r^2} u(e^{i\phi}) \, d\phi. \]
That is, we are integrating the values of \( u \) on the circle with the Poisson kernel.
(Hint. Use the Cauchy integral formula on \( f = u + iv \), applied to a disk of radius slightly less than 1. Then take a limit.)