

# MATH 309 - Solution to Homework 1

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1. We can parametrize the torus as the following set of points in  $\mathbb{R}^3$ :

$$T := \{((2 + \cos(\theta)) \cos(\varphi), (2 + \cos(\theta)) \sin(\varphi), \sin(\theta)) : \theta \in [0, 2\pi], \varphi \in [0, 2\pi]\},$$

Where  $\theta$  is the angle coordinate “inside the tube” and  $\varphi$  describes which slice of the torus we are on, i.e.  $\varphi$  is the polar angle for the projection onto the  $xy$ -plane of a point on the torus (see figure 1 below). Now if  $p, q$  are coprime, then the parametrized curve  $((2 + \cos(pt)) \cos(qt), (2 + \cos(pt)) \sin(qt), \sin(pt))$  is a knot which lies on the torus and winds  $p$  times around the meridian and  $q$  times around the longitude (again, see figure 1).

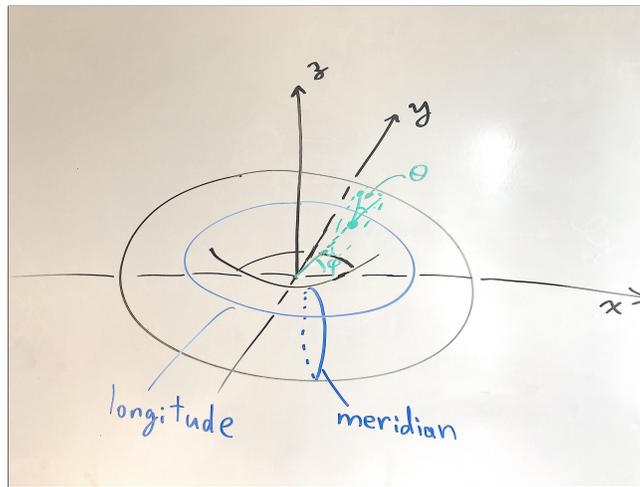


Figure 1: The Torus with  $\theta, \varphi$ , the meridian and longitude indicated

- (a) If  $p = 1$  and  $q = 2$ , then the knot is a curve which wraps once around the meridian and twice around the longitude: see Figure 2. This knot is trivial: a Reidemeister move of type I turns it

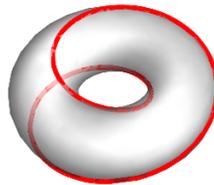


Figure 2: The  $p = 1, q = 2$  knot.

into the unknot.

- (b) The  $p = 2, q = 3$  knot wraps twice around the meridian and 3 times around the longitude, and the  $p = 3, q = 2$  knot wraps 3 times around the meridian and 2 times around the longitude. These two knots are pictured in Figure 3 below, on the torus. The Reidemeister moves to go from the projection of the  $p = 2, q = 3$  knot to the  $p = 3, q = 2$  knot appear in Figure 4.

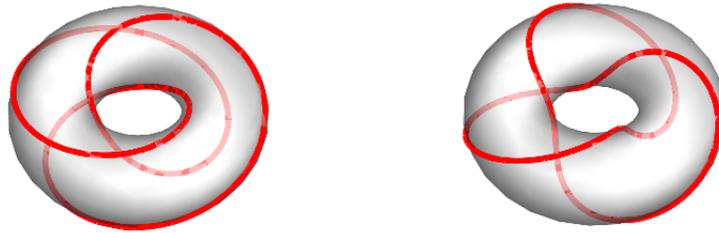


Figure 3: Left: the  $p = 2, q = 3$  knot. Right: the  $p = 3, q = 2$  knot.

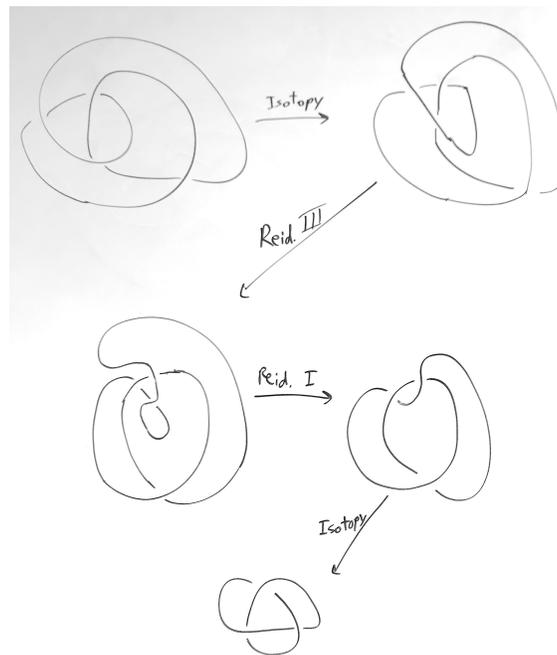


Figure 4: Moves to get from the  $p = 2, q = 3$  knot to the  $p = 3, q = 2$  knot

2. This is proven at the beginning of section 3.1 in Adams. The idea is to travel along the knot and deal with each crossing one by one, as you meet it on your path so that you are always going down hill. So pick a point  $P_0$  on the knot and a direction. Travelling along the knot, do the following at every crossing that you encounter: if this is the first time you arrive at this crossing, make sure that you go over it (by changing it or not) and if it is not the first time, do nothing and keep travelling along the knot; stop travelling once you reach  $P_0$  again. Here is a slightly different argument from Adams':

*Proof.* Consider running the algorithm and halting right before the first time you have to go *under* a crossing. At this stage, the part of the knot that you travelled on lies above the rest of the knot, hence it can be deformed into a small loop that misses the rest of the knot, as in Figure 5 below. We can get rid of this crossing by a Reidemeister move of type I. If we now continue running the algorithm after this Reidemeister move, the output is the same as if we had not done the move (to prove this, there are two cases to consider: meeting a new crossing or meeting a crossing for a second time). Thus running the algorithm until we reach an undercrossing gives us the same result as running the algorithm on a new knot with fewer crossings and where we did not yet reach an undercrossing. Since a knot has

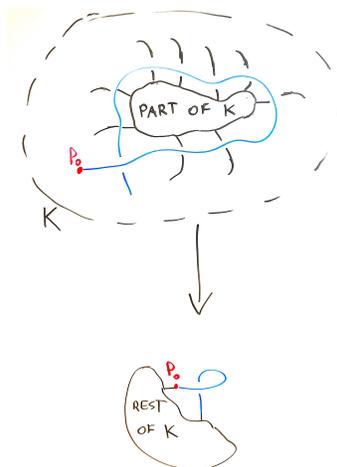


Figure 5: Deforming a piece of the knot into a small loop

finitely many crossings, this proves that the algorithm lets us get rid of all of them, which is to say that the algorithm turns every knot into the unknot.  $\square$

3. **Definition:** A link is Brunnian if it is nontrivial, but becomes trivial after the removal of any component, i.e. removing a component turns the link into a number of unknots which are not linked.

(a) Let's look at the Borromean rings, which form a Brunnian link with 3 components. In the top of Figure 6, we reorganize it to look like a braid (this is a precise mathematical term, but do not worry about that) and then pull the blue and green ropes tight so that only the red rope is winding around them. The pattern observed in the Borromean rings may give you an idea for how to construct a Brunnian link with more components, as we do at the bottom of Figure 6.

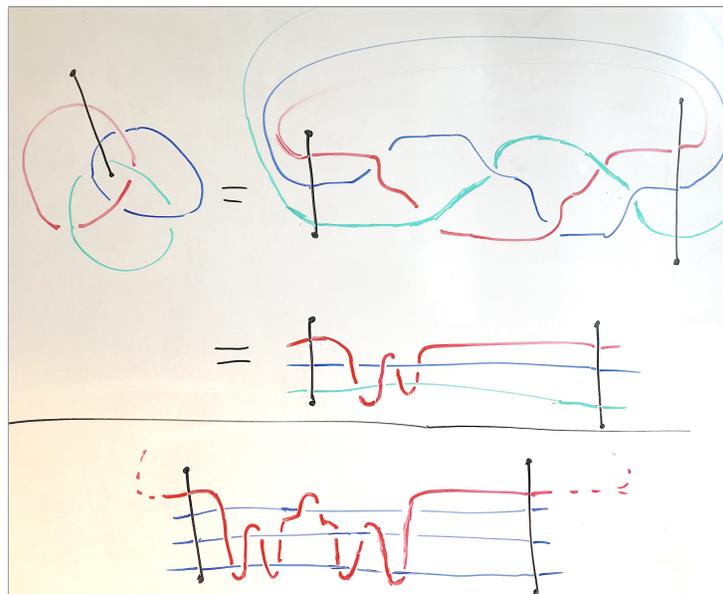


Figure 6: Top: the Borromean rings, pulled into a braid and rearranged. Bottom: a 4-component Brunnian link. Note that the ends of the braid (left and right of the vertical black lines) should be connected to form an honest link.

- (b) Part (a) gives a hint for a general procedure. Let us call the braided part of the link (i.e. the one sandwiched between the two vertical black lines) the **interesting part** of the link. In Figure 7 we construct a 5-component Brunnian link. The following proof should be read while looking at the picture.

Note that the interesting part of the link in Figure 7 is made up of two copies of the interesting part of the 4-component Brunnian link from Figure 6, which we may label  $B_4$ . It is important to note that the second copy of  $B_4$  is mirrored! This ensures that if we remove the top blue rope, then we can undo the twisting of the red rope in order to obtain a trivial link with 4 components. If we remove any other blue rope, then we can undo the twisting of the red rope inside of each copy of  $B_4$  to again obtain a trivial link of 4 components. Finally, it is clear that removing the red rope also gives a trivial link. This gives the procedure for constructing a Brunnian link with arbitrarily many components: just replace "4" with  $n$  in the bottom picture and have  $n - 1$  blue strands going into the interesting part  $B_n$ .

**Remark.** The proof just given is by induction: we proved that if we have a Brunnian link with  $n - 1$  components, then we can construct one with  $n$  components, as long as  $n - 1 \geq 3$ . The base case holds because we know how to make a Brunnian link with 3 components, but we could also have the base case be a Brunnian link with 2 components. Trying to run the proof starting with 1 or 0 component links would get us in trouble, since every knot is a Brunnian link according to the definition. Maybe the definition should also stipulate that the link should have at least 2 components.

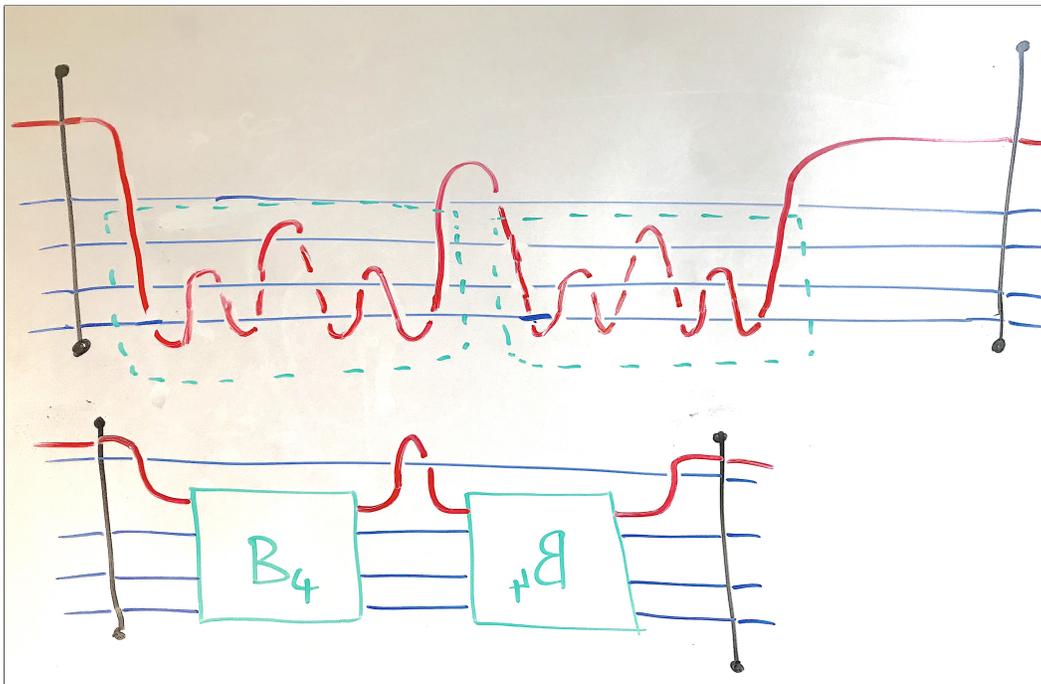


Figure 7: A 5-component Brunnian link and a description of it which generalizes to construct  $n$ -component Brunnian links, for all  $n \geq 2$

4.  $6_1$  is tricolourable, whereas  $6_2$  and  $6_3$  are not. See Figure 8 below. One way to see that  $6_1$  and  $6_2$  cannot be tricoloured is that each of the chosen knot projections for  $6_2$  and  $6_3$  has a strand which appears next to every other strand of the knot, at some crossing. Let's call such a strand a **friendly strand**. This strand is coloured red and pointed at in the picture. Therefore, in order to colour these knots, we would have to use only two colours on all the other strands. It is easy to check that this cannot be done for these projections. Thus  $6_2$  and  $6_3$  are not tricolourable.

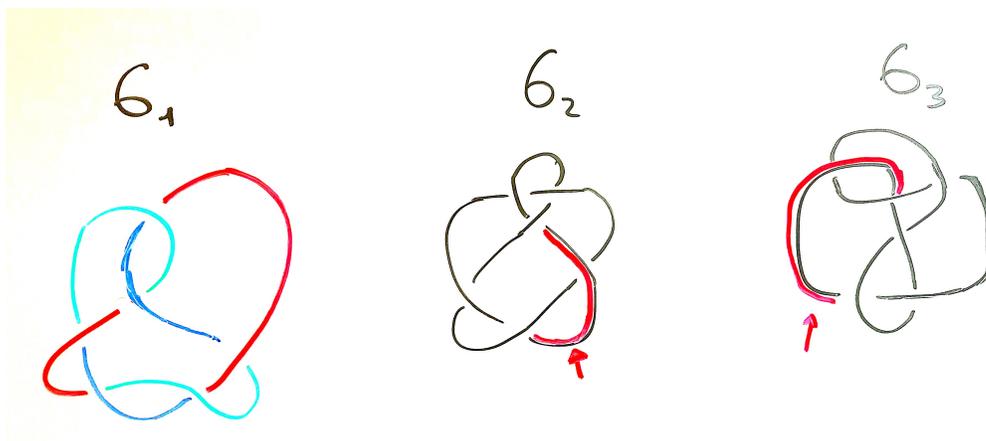


Figure 8: The knots  $6_1$ ,  $6_2$  and  $6_3$ . In each of  $6_2$  and  $6_3$ , the indicated strand appears in every crossing.

**Remark.** You might be tempted to guess that if a knot has a friendly strand, then it cannot be tricoloured, but this is not the case! The usual projection of the trefoil knot has a friendly strand and the trefoil is tricolourable. You may then be tempted to guess that if a knot has a friendly strand and at least 4 crossings, then it cannot be tricoloured, but this is also not true! See the (2,3) knot (which is a projection of the trefoil) from question 1. I am not sure what the correct general statement should be.

5. The type III Reidemeister move preserves tricolourability.

*Proof.* We will prove this statement for one of the two type III Reidemeister moves. The proof for the second type III move is almost identical. The direct way to prove this is by checking all possible cases, like doing a Sudoku (it turns out that there are only 4 in each direction). To begin, let's label the 6 strands which appear in the definition of the type III move, as in Figure 9. Each of the labels  $a, b, c, d, f, x, y$  stands for a colour in the set  $\{\text{Red, Blue, Green}\}$ . We will prove first that if the picture

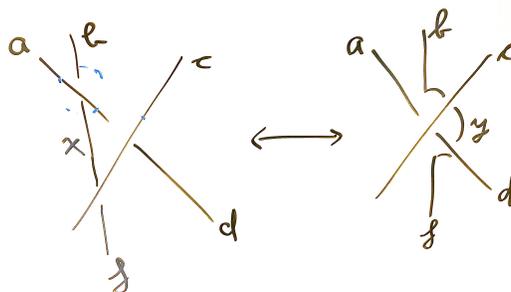


Figure 9: Reidemeister III move with labelled strands

on the left is tricoloured, then we can always make  $y$  be a colour that makes the picture on the right tricoloured. We will also prove that if the picture on the right is tricoloured, then we can pick a colour for  $x$  that makes the picture on the left tricoloured.

The main idea for the proof is to recall that the 3 strands meeting at a crossing can only have distinct or identical colours (i.e. it is not possible to have two strands of the same colour and one with a different colour) and, in case of confusion, to draw a picture in each subcase (which is not done here).

- (a) Suppose first that  $a, b, c, d, f, x$  are a consistent tricolouring of the left-hand picture. We have two cases to consider:
- i. Case 1:  $x = a = b$ . WLOG, let us call this colour Red. We consider two subcases:
    - A. Subcase 1.1:  $x = f = c$ . In this case all strands on the left are Red and we can make  $y$  be red as well.
    - B. Subcase 1.2:  $x, f, c$  are distinct. So,  $c$  is not Red. We can conclude that  $f = d$ , since both  $f$  and  $d$  must be different from  $c$  and cannot be Red. We can then make  $y$  be the same colour as  $d$ .
  - ii. Case 2:  $a, b, x$  are distinct colours.
    - A. Subcase 2.1:  $x = f = c$ . In this case we can make  $y$  the same colour as  $a$ , since both  $a$  and  $y$  have to be a different colour from  $c$  and  $d$  (which are not the same colour).
    - B. Subcase 2.2:  $x, f, c$  are distinct. Since both  $c$  and  $b$  have to be different from  $a$  and  $x$  (which are different colours), we can conclude  $c = b$ . Similarly, we may conclude that  $f = a$  and  $d = x$ , and thus we can colour  $y$  the same colour as  $a$ .
- (b) Suppose now that  $a, b, c, d, f, y$  are a consistent tricolouring for the right-hand picture. We will now just say what we can conclude about the colours and leave the reader to provide the "Sudoku"-like argument.
- i. Case 1:  $b = c = y = \text{Red}$ , say.
    - A. Subcase 1.1:  $d = f = y$ . Therefore all the strands are Red and we can make  $x$  Red as well.
    - B. Subcase 1.2:  $d, f, y$  are distinct. Therefore  $a = f$  and we can give  $x$  the same colour as  $d$ .
  - ii. Case 2:  $b, c, y$  are distinct.
    - A. Subcase 2.1:  $y = d = f$ . Conclude that  $a = b$ , therefore  $x = a$  is consistent.
    - B. Subcase 2.2:  $y, d, f$  are distinct. Conclude that  $b = d$  and  $a = y$ , therefore we can colour  $x = b$ .

□

6. The trefoil knot is tricolourable, whereas the figure-eight is not. This proves that they are distinct knots. To prove that the figure-eight knot is not tricolourable, we can just show that the projection in Figure 10 is not tricolourable. It is easy because the figure-eight knot has two friendly strands, forcing us to colour all other strands using 1 single colour, that is not blue or red. This cannot be done for this projection. By Reidemeister's theorem and the fact that tricolourations are left invariant by Reidemeister moves, the figure-eight knot is distinct from the trefoil knot.

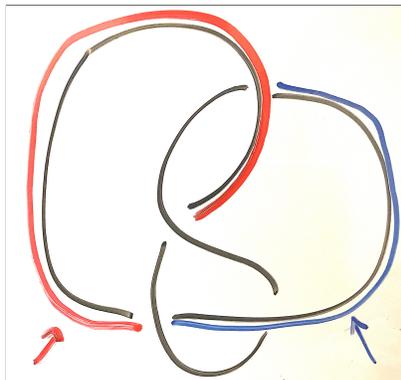


Figure 10: Both indicated stands are friendly.

7. (a) It is easy to show that Reidemeister moves of type I and II preserve the labelling scheme. Here is the proof that Reidemeister moves of type III preserve it. Suppose first we are in position to apply a type III move to a knot crossing where the strands are labelled using  $a, b, c, d, f, x$ , as in the left side of Figure 9. The three crossings in the initial labelling imply that the following equations are satisfied (mod 5):

$$2a = b + x$$

$$2c = f + x$$

$$2c = a + d$$

Using the first and third equations to eliminate the variables  $x$  and  $a$ , we obtain

$$2c - b = 2d - f.$$

Now the one of the two crossings on the right-hand side where strand  $y$  appears implies that  $y = 2c - b$ , whereas the other crossing implies that  $y = 2d - f$ . Since these are in fact the same, we can coherently label every strand by making  $y = 2a - f$ .

To go the other way, that is to show that given the labelling on the right-hand side of Figure 9, we can label  $x$  coherently, we do the same thing. The right hand labelling implies that  $a, b, c, d, f, y$  must satisfy

$$2c = b + y$$

$$2d = f + y$$

$$2c = a + d$$

Using the first and last equations to eliminate  $y$  and  $d$ , we obtain  $2a - b = 2c - f$ , so we may use this number to label  $x$ .

- (b) Tricolourability is a labelling of the strands using  $\{0, 1, 2\}$  subject to the relation  $x + y - 2z \equiv 0 \pmod{3}$ . This is the same as the equation  $x + y + z \equiv 0 \pmod{3}$ .