1. (5.42) Show that there exist two distinct irrational numbers $a$ and $b$ such that $a^b$ is rational.

**Solution:** Consider the irrational numbers $\sqrt{3}$ and $\sqrt{2}$. If $\sqrt{3} \sqrt{2}$ is rational, then $a = \sqrt{3}$ and $b = \sqrt{2}$ have the desired properties. On the other hand, if $\sqrt{3} \sqrt{2}$ is irrational, then $\sqrt{3} \sqrt{2}^2 = \sqrt{3} \cdot 2 = 3$ is rational. Thus $a = \sqrt{3} \sqrt{2}$ and $b = \sqrt{2}$ have the desired properties. □

2. (5.46)
(a) Prove that there exist four distinct positive integers such that each integer divides the sum of the remaining integers.

**Solution:**

**Proof:** Observe that the integers 1, 2, 3 and 6 have the desired properties, since they are four distinct positive integers, and $1|2+3+6$, $2|(1+3+6)$, $3|(1+2+6)$ and $6|(1+2+3)$. □

(b) The problem in (a) should suggest another problem to you. State and solve such a problem.

**Solution:** The related problem is: “Prove that there exist five distinct positive integers such that each integer divides the sum of the remaining integers.”

**Proof:** Observe that the integers 1, 2, 3, 6 and 12 have the desired properties, since they are five distinct positive integers, and $1|2+3+6+12$, $2|(1+3+6+12)$, $3|(1+2+6+12)$, $6|(1+2+3+12)$ and $12|(1+2+3+6)$. □

3. (5.50) Disprove the statement: There is a real number $x$ such that $x^6 + x^4 + 1 = 2x^2$.

**Solution:** We disprove this statement by proving that for all real numbers $x$, $x^6 + x^4 + 1 \neq 2x^2$.

**Proof:** (of the negation) Let $x \in \mathbb{R}$. Observe that

$$x^6 + x^4 - 2x^2 + 1 = x^6 + (x^2 - 1)^2.$$  

Since $x^6 \geq 0$ and $(x^2 - 1)^2 \geq 0$ for any $x \in \mathbb{R}$, $x^6 + (x^2 - 1)^2 = 0$ if and only if $x^6 = 0$ and $(x^2 - 1)^2 = 0$. However, $x^6 = 0$ if and only if $x = 0$, whereas $(x^2 - 1)^2 = 0$ if and only if $x = -1$ or $x = 1$. It is therefore impossible to simultaneously have $x^6 = 0$ and $(x^2 - 1)^2 = 0$. Hence, there is no real number $x$ such that $x^6 + (x^2 - 1)^2 = 0$. Thus

$$x^6 + x^4 - 2x^2 + 1 = x^6 + (x^2 - 1)^2 \neq 0$$

and so, for all $x \in \mathbb{R}$, $x^6 + x^4 - 2x^2 + 1 \neq 0$. □
4. (5.52) The integers 1, 2, 3 have the property that each divides the sum of the other two. Indeed, for each positive integers \( a \), the integers \( a, 2a, 3a \) have the property that each divides the sum of the other two. Show that the following statement is false.

There exists an example of three distinct positive integers different from \( a, 2a, 3a \) for some \( a \in \mathbb{N} \) have the property that each divides the sum of the other two.

**Solution:** We prove this statement is false by proving that its negation “any three distinct positive integers with the property that each divides the sum of the other two can be written in the form \( a, 2a, 3a \) for some \( a \in \mathbb{N} \)” is true.

Let \( a, b, c \) be positive integers with \( a < b < c \) such that each of \( a, b, c \) divides the sum of the other two. Since \( a | (b + c) \), \( b | (a + c) \) and \( c | (a + b) \), it follows that there exist positive integers \( r, s, t \) such that \( b + c = ra \), \( a + c = sb \) and \( a + b = tc \).

Since \( a < b < c \), it follows that \( ra = b + c > 2a \), so \( r \geq 3 \). Similarly, since \( a < b < c \), \( sa < sb < sc \) and so \( sb = a + c > sa \) so \( c > a(s - 1) \) and since \( c > a \) this requires \( s \geq 2 \). Since \( tc = a + b < c + c = 2c \), it follows that \( t = 1 \) and so \( c = a + b \).

Now \( sb = a + c = a + (a + b) = 2a + b \), which implies that \( (s - 1)b = 2a < 2b \). Therefore, \( s \leq 2 \), which implies that \( s = 2 \). Therefore, \( b = 2a \) and \( c = a + b = 3a \). Thus, we have \( b = 2a \) and \( c = 3a \), so the set of integers \( a, b, c \) can be written in the form \( a, 2a, 3a \), proving the statement.

5. (7.4) It has been stated that the German mathematician Christian Goldbach is known for a conjecture he made concerning primes. We refer to this conjecture as Conjecture A.

**Conjecture A:** Every even integer at least 4 is the sum of two primes

Goldbach made two other conjectures concerning primes.

**Conjecture B:** Every integer at least 6 is the sum of three primes

**Conjecture C:** Every odd integer at least 9 is the sum of three odd primes

Prove that the truth of one or more of these three conjectures implies the truth of one or two of the other conjectures.

**Solution:** This problem is really open and you can choose many different things to prove here. I will prove that “if Conjecture A is true, then both Conjecture B and Conjecture C are also true.”

**Proof:**

Suppose that Conjecture A is true. We prove two things: (I) Conjecture B is true and (II) Conjecture C is true.

**Proof of (I)**

Let \( n \) be an integer greater than or equal to 6. Then \( n - 2 \) is an integer greater than or equal to 4. Hence, by conjecture A, we have \( n - 2 = p_1 + p_2 \) where \( p_1 \) and \( p_2 \) are primes. Thus

\[
 n = p_1 + p_2 + 2
\]
Since 2, \( p_1 \) and \( p_2 \) are all prime numbers, conjecture B is true.

**Proof of (II)**

Let \( n \) be an odd integer greater than or equal to 9. Then \( n - 5 \) is an integer greater than or equal to 4. Hence, by conjecture A, we have \( n - 5 = p_1 + p_2 \) where \( p_1 \) and \( p_2 \) are primes. We now have \( n = p_1 + p_2 + 5 \). Note that since \( n \) is odd, \( p_1 + p_2 \) must be even, and so \( p_1 \) and \( p_2 \) must have the same parity. We now consider two cases.

**Case 1:** If \( p_1 \) and \( p_2 \) are both even, then since they are prime we must have \( p_1 = 2 = p_2 \). Hence \( n = 2 + 2 + 5 = 3 + 3 + 3 \) and so \( n \) is the sum of three odd primes.

**Case 2:** If \( p_1 \) and \( p_2 \) are both odd, then we have \( n = p_1 + p_2 + 5 \) and so \( n \) is the sum of three odd primes.

In all cases, \( n \) can be written as the sum of three odd primes, and so conjecture C is true. \( \square \)

6. (7.16)

(a) Express the following quantified statement in symbols:

\[ \text{for every integer } n, \text{ there exists an integer } m \text{ such that } (n - 2)(m - 2) > 0. \]

**Solution:**

\[ \forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ s.t. } (n - 2)(m - 2) > 0 \]

(b) Express in symbols the negation of the statement in (a).

**Solution:**

\[ \exists n \in \mathbb{Z} \text{ s.t. } \forall m \in \mathbb{Z}, (n - 2)(m - 2) \leq 0 \]

(c) Show that the statement in (a) is false.

**Solution:** We prove that the statement in (b) is true. Consider \( n = 2 \). Then for any \( m \in \mathbb{Z} \) we have \( (n - 2)(m - 2) = 0 \cdot (m - 2) = 0 \), and so the statement in (b) is true. Thus, the original statement in (a) is false.

7. (7.22)

(a) Express the following quantified statement in symbols:

\[ \text{There exist two integers } a \text{ and } b \text{ such that for every positive integer } n, \ a < \frac{1}{n} < b. \]

**Solution:**

\[ \exists a, b \in \mathbb{Z} \text{ s.t. } \forall n \in \mathbb{Z}, n > 0 \Rightarrow a < \frac{1}{n} < b \]
(b) Prove that the statement in (a) is true.

**Solution:** Consider \( a = 0 \) and \( b = 2 \). Let \( n \in \mathbb{Z} \) and suppose \( n > 0 \). Then we have \( n \geq 1 \), and hence \( \frac{1}{n} \leq 1 \). Since we also have \( n > 0 \), it is also true that \( 0 < \frac{1}{n} \). Hence, \( 0 < \frac{1}{n} \leq 1 \). Thus, \( 0 < \frac{1}{n} < 2 \) and so \( a < \frac{1}{n} < b \). \( \square \)

8. (7.26) Prove the following statement. For every positive real number \( a \) and positive rational number \( b \), there exist a real number \( c \) and irrational number \( d \) such that \( ac + bd = 1 \).

**Solution:**

**Proof:**
We will need the following lemma (exercise 5.17 in the text): “when an irrational number is divided by a (non-zero) rational number, the resulting number is irrational.”

**Proof of Lemma:**
Assume, to the contrary, that there exist an irrational number \( t \) and a nonzero rational number \( u \) such that \( t/u \) is a rational number. Then \( t/u = p/q \) for some integers \( p \) and \( q \), with \( p, q \neq 0 \). Since \( u \) is a nonzero rational number, \( u = r/s \) for some \( r, s \in \mathbb{Z} \) where \( r, s \neq 0 \). Thus

\[
    t = \frac{up}{q} = \frac{rp}{sq}
\]

Since \( rp, sq \in \mathbb{Z} \) and \( sq \neq 0 \), it follows that \( t \) is a rational number, which is a contradiction to our assumption that \( t \) is irrational. Hence, the lemma is proved.

We now prove the main result. Let \( a \) be a positive real number and \( b \) be a positive rational number. Consider the numbers \( c = \frac{(1 - \sqrt{2})}{a} \) and \( d = \frac{\sqrt{2}}{b} \). Since \( a \) is a non-zero real number, \( c \) is a real number. Also, since \( \sqrt{2} \) is irrational and \( b \) is a non-zero rational number, by the lemma we have that \( d \) is an irrational number. Finally,

\[
    ac + bd = a\left(\frac{1 - \sqrt{2}}{a}\right) + b\frac{\sqrt{2}}{b} = 1 - \sqrt{2} + \sqrt{2} = 1
\]

and so the statement is proved. \( \square \)

9. (7.64) Prove or disprove: There exist an irrational number \( a \) and a rational number \( b \) such that \( a^b \) is irrational.

**Solution:**

**Proof:** Consider \( a = \sqrt{2} \) and \( b = 1 \). We have already proven that \( \sqrt{2} \) is irrational, and it is known that \( 1 \) is a rational number. Also, \( a^b = \sqrt{2}^1 = \sqrt{2} \), which is irrational. \( \square \)

10. (7.70) Prove or disprove: Let \( A, B \) and \( C \) be sets. Then \( A \cup (B - C) = (A \cup B) - (A \cup C) \).

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Solution:

Disproof: We give a counterexample. Consider $A = \{1\}$, $B = \emptyset$, $C = \emptyset$. Then we have $A \cup (B - C) = \{1\} \cup \emptyset = \{1\}$, and $(A \cup B) - (A \cup C) = \{1\} - \{1\} = \emptyset$. Hence, $A \cup (B - C) \neq (A \cup B) - (A \cup C)$.

11. (7.74) Prove or disprove: Let $a$, $b$, $c \in \mathbb{Z}$. Then at least one of the numbers $a + b$, $a + c$ and $b + c$ is even.

Solution:

Proof: Suppose, for a contradiction, that there exist $a, b, c \in \mathbb{Z}$ such that $a + b$, $a + c$ and $b + c$ are all odd. Then there exist integers $k, l, m$ such that $a + b = 2k + 1$, $a + c = 2l + 1$ and $b + c = 2m + 1$. If we add these all together, we have

$$(a + b) + (a + c) + (b + c) = 2k + 1 + 2l + 1 + 2m + 1$$

Hence,

$$2(a + b + c) = 2(k + l + m + 1) + 1$$

Since $a + b + c$ is an integer, $2(a + b + c)$ is even. Since $k + l + m + 1$ is an integer, $2(k + l + m + 1) + 1$ is odd. Thus, the equation above, $2(a + b + c) = 2(k + l + m + 1) + 1$ is a contradiction. Hence, the original statement must be true. □

12. (7.78) Prove or disprove: For every odd prime $p$, there exist positive integers $a$ and $b$ such that $a^2 - b^2 = p$.

Solution:

Proof: Let $p$ be an odd prime. Then $p = 2k + 1$ for some $k \in \mathbb{N}$. Consider $a = k + 1$ and $b = k$. Then $a$ and $b$ are positive integers and we have

$$a^2 - b^2 = (k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1 = p.$$ □

13. (8.4) Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then $R_1 = \{(a, 2), (a, 3), (b, 1), (b, 3), (c, 4)\}$ is a relation from $A$ to $B$, which $R_2 = \{(1, b), (1, c), (2, a), (2, b), (3, c), (4, a), (4, c)\}$ is a relation from $B$ to $A$. A relation $R$ is defined on $A$ by $xRy$ if there exists $z \in B$ such the $xR_1z$ and $zR_2y$. Express $R$ by listing its elements.

Solution:

$$R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, c)\}$$