1. (3.16) Let $x \in \mathbb{Z}$. Prove that if $7x + 5$ is odd, then $x$ is even.

**Solution:** We will prove the contrapositive:

“If $x$ is odd, then $7x + 5$ is even”

Assume that $x$ is odd. Then $x = 2n + 1$ for some integer $n$. Therefore,

$$7x + 5 = 7(2n + 1) + 5$$
$$= 14n + 12$$
$$= 2(7n + 6)$$

Since $7n + 6$ is an integer, we have that $7x + 5$ is even and the contrapositive is proved. Hence, the original statement is also true. □

2. (3.18) Let $x \in \mathbb{Z}$. Prove that $5x - 11$ is even if and only if $x$ is odd.

**Solution:** Here we must prove the two implications:

1. If $5x - 11$ is even, then $x$ is odd.
2. If $x$ is odd, then $5x - 11$ is even.

**Proof of 1.**
We will prove the contrapositive of statement 1.

“If $x$ is even, then $5x - 11$ is odd”

Assume that $x$ is even. Then there exists $n \in \mathbb{Z}$ such that $x = 2n$, and we have

$$5x - 11 = 5(2n) - 11 = 10n - 11 = 2(5n - 6) + 1.$$ 

Since $5n - 6$ is an integer, this shows that $5x - 11$ is odd and the contrapositive is proved, as is the original statement.

**Proof of 2.**
Assume that $x$ is odd. Then there exists $n \in \mathbb{Z}$ such that $x = 2n + 1$, and we have

$$5x - 11 = 5(2n + 1) - 11 = 10n - 6 = 2(5n - 3)$$

Since $5n - 3$ is an integer, $5x - 11$ is even.

Since we have proved both required implications, we have proved that $5x - 11$ is even if and only if $x$ is odd. □
3. (3.20) Let \( x \in \mathbb{Z} \). Prove that \( 3x + 1 \) is even if and only if \( 5x - 2 \) is odd.

**Solution:**
To prove this, we will first prove the following lemma.

**Lemma:** Let \( x \in \mathbb{Z} \). Then \( 3x + 1 \) is even if and only if \( x \) is odd.

**Proof of lemma:**

We must prove the two statements: (1) “if \( x \) is odd, then \( 3x + 1 \) is even” and (2) “if \( 3x + 1 \) is even, then \( x \) is odd.”

To prove (1), suppose that \( x \) is odd. Then \( x = 2n + 1 \) for some integer \( n \), and so \( 3x + 1 = 3(2n + 1) + 1 = 2(3n + 2) \). Since \( 3n + 2 \) is an integer, we have shown that \( 3x + 1 \) is even.

We now prove the contrapositive of (2), that is “if \( x \) is even, then \( 3x + 1 \) is odd.” Suppose that \( x \) is even. Then \( x = 2m \) for some integer \( m \) and so \( 3x + 1 = 3(2m) + 1 = 2(3m) + 1 \). Since \( 3m \) is an integer, we have shown that \( 3x + 1 \) is odd.

We now prove the statement in the problem.

**Proof:** To prove this biconditional, we need to prove the two component statements, (i) “if \( 5x - 2 \) is odd, then \( 3x + 1 \) is even” and (ii) “if \( 3x + 1 \) is even, then \( 5x - 2 \) is odd”, but let us first start by proving the lemma “\( 3x + 1 \) is even if and only if \( x \) is odd.”

To prove (i), we instead prove the contrapositive “if \( 3x + 1 \) is odd, then \( 5x - 2 \) is even.” Suppose that \( 3x + 1 \) is odd. Then by the lemma, \( x \) is even, and so \( x = 2m \) for some integer \( m \). Hence, we have that \( 5x - 2 = 5(2m) - 2 = 2(5m - 1) \) and since \( 5m - 1 \) is an integer we have that \( 5x - 2 \) is even.

To prove (ii), suppose that \( 3x - 1 \) is even. Then by the lemma \( x \) is odd and so there exists an integer \( k \) such that \( x = 2k + 1 \). We therefore have that \( 5x - 2 = 5(2k + 1) - 2 = 10k + 3 = 2(5k + 1) + 1 \). Since \( 5k + 1 \) is an integer, this shows that \( 5k - 2 \) is odd.

Since we have proved (i) and (ii), we have proved the stated biconditional. \( \square \)

4. (3.28) Let \( x, y \in \mathbb{Z} \). Prove that if \( xy \) is odd, then \( x \) and \( y \) are odd.

**Solution:**

**Proof:**

There are three possible cases:

Case (1) \( x \) is even: In this case there exists an integer \( n \) such that \( x = 2n \) and so we have \( xy = (2n)y = 2(ny) \) and since \( ny \) is an integer, we have that \( xy \) is even. In this case the hypothesis of the implication is false, and so the implication is true.
Case (2) $x$ is odd and $y$ is even: In this case there exists an integer $n$ such that $y = 2n$ and so we have $xy = x(2n) = 2(nx)$ and since $nx$ is an integer, we have that $xy$ is even. In this case the hypothesis of the implication is false, and so the implication is true.

Case (3) $x$ is odd and $y$ is odd: In this case the conclusion of the implication is true, and so the implication is true.

We have shown that the given statement is true for all possible cases for $x$ and $y$, and so the statement is proved. □

5. (3.32)

(a) Let $x$ and $y$ be integers. Prove that $(x + y)^2$ is even if and only if $x$ and $y$ are of the same parity.

**Solution:** There are three possible cases:

Case (1) $x$ and $y$ are both even: Then there exist integers $n$ and $m$ such that $x = 2m$ and $y = 2n$ and so we have $(x + y)^2 = (2m + 2n)^2 = 4(m + n)^2 = 2(2(m + n)^2)$. Since $2(m + n)^2$ is an integer, this shows that $(x + y)^2$ is even. Hence, in this case both statements $x$ and $y$ are of the same parity and $(x + y)^2$ is even are true.

Case (2) $x$ and $y$ are both odd: Then there exist integers $n$ and $m$ such that $x = 2m + 1$ and $y = 2n + 1$ and so we have $(x + y)^2 = (2m + 2n + 2)^2 = 4(m + n + 1)^2 = 2(2(m + n + 1)^2)$. Since $2(m + n + 1)^2$ is an integer, this shows that $(x + y)^2$ is even. Hence, in this case both statements $x$ and $y$ are of the same parity and $(x + y)^2$ is even are true.

Case (3) $x$ and $y$ have opposite parity: Without loss of generality, assume that $x$ is even and $y$ is odd. Then there exist integers $n$ and $m$ such that $x = 2m$ and $y = 2n + 1$ and so we have $(x + y)^2 = (2m + 2n + 2)^2 = 4m^2 + 4n^2 + 8mn + 4m + 4n + 1 = 2(2m^2 + 2n^2 + 4mn + 2m + 2n) + 1$ and since $2m^2 + 2n^2 + 4mn + 2m + 2n$ is an integer, this shows that $(x + y)^2$ is odd. Hence, in this both statements $x$ and $y$ are of the same parity and $(x + y)^2$ is even are false.

We have shown that for all possible cases of $x$ and $y$ the component statements of the biconditional have the same truth value. Hence, the biconditional is always true and the statement is proved. □

(b) Restate the result in (a) in terms of odd integers.

**Solution:** We can restate this in terms of odd integers by taking the contrapositive of each of the component implications.

“(x + y)^2 is odd if and only if x and y have different parity”