Trigonometric integrals $\int \tan^m x \sec^n x \, dx$

General approach

1. Is $m$ odd?
   - yes
     - $m$ is odd
       - Use $\tan x = \frac{\sin x}{\cos x}$
         and $\sec x = \frac{1}{\cos x}$
   - no
     - Is $n$ odd?
       - yes
         - $m$ is even, $n$ is odd
         - Beyond the scope of MATH101
       - no
         - Is $n \geq 2$?
           - yes
             - $m$ is even, $n$ is even, $n \geq 2$
               - 1. Separate one $\sec^2 x$
               - 2. All other $\sec x \rightarrow \tan x$
                 using $\sec^2 x = 1 + \tan^2 x$
               - 3. Substitution
                 $u = \tan x$
           - no
             - $m$ is even, $n = 0$
               - 1. Separate one $\tan^2 x$
               - 2. Simplify
               - 3. One integral (that contains $\sec^2 x$)
                 $u = \tan x$
                 second integral-repeat 1-3 if needed
Case m is even, n = 0.

\[ m = 2k. \]

\[
\int \tan^{2k} x \, dx = \int \tan^{2k-2} x \tan^2 x \, dx
\]
\[ = \int \tan^{2k-2} x (\sec^2 x - 1) \, dx \]
\[ = \int \tan^{2k-2} x \sec^2 x \, dx - \int \tan^{2k-2} \, dx \]

For this integral,

use \( u = \tan x \)

Example \( \int \tan^2 x \, dx \)

Solution Using \( \tan^2 x = \sec^2 x - 1 \),

\[
\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C
\]

Example \( \int \tan^4 x \, dx \)

Solution Using \( \tan^2 x = \sec^2 x - 1 \),

\[
\int \tan^4 x \, dx = \int \tan^2 x \tan^2 x \, dx = \int \tan^2 x (\sec^2 x - 1) \, dx
\]
\[ = \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \]

For first integral:

\[ \left\{ \begin{array}{l}
    \text{let } u = \tan x \\
    u' = \sec^2 x
  \end{array} \right. \]

\[ = \int u^2 \, du - (\tan x - x) + C \]
\[ = \frac{u^3}{3} - (\tan x - x) + C \]
\[ = \frac{\tan^3 x}{3} - \tan x + x + C \]