Integration by parts

Before we used the chain rule for derivatives to obtain the substitution rule for integrals. Now any use of the product rule for derivatives to evaluating integrals?

Recall The product rule ($u(x)$ and $v(x)$ are differentiable)

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x).$$

Getting back to integrals Integrate both sides.

$$\int [u(x)v(x)]' dx = \int (u'(x)v(x) + u(x)v'(x)) dx$$

The LHS = $u(x)v(x) + C$

$$\int u'v dx + \int uv' dx = uv + C$$

$$\int uv' dx = uv - \int u'v dx$$

$$\int_{a}^{b} [u(x)v(x)]' dx = \int_{a}^{b} (u'(x)v(x) + u(x)v'(x)) dx$$

The LHS = $u(b)v(b) - u(a)v(a)$

$$\int_{a}^{b} uv' dx = u(b)v(b) - u(a)v(a) - \int_{a}^{b} u'v dx$$
Integration by parts

Theorem (integration by parts)

Let \( u(x) \) and \( v(x) \) be continuously differentiable, i.e., \( u'(x) \) and \( v'(x) \) exist and are continuous. Then

\[
\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx
\]

and

\[
\int_{a}^{b} u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_{a}^{b} u'(x)v(x)\,dx
\]

The application of these formulas is known as integration by parts.

Example: Compute \( \int x \log x\,dx \)

Solution: \( \int x \log x\,dx = \frac{x^2}{2} \log x - \int x \frac{1}{x} \,dx \)

\[
= \frac{x^2}{2} \log x - \frac{1}{2} \int x \,dx
\]

\[
= \frac{x^2}{2} \log x - \frac{1}{2} \left( \frac{x^2}{2} \right) + C
\]

\[
= \frac{x^2}{2} \log x - \frac{x^2}{4} + C.
\]
Integration by parts

Example

Compute $\int x \cos x \, dx$

Choice 1:

$u = x \quad u' = 1$

$v' = \cos x \quad v = \sin x$

By the integration by parts,

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x - (-\cos x) + C$$

$$= x \sin x + \cos x + C$$

Choice 2:

$u = \cos x \quad u' = -\sin x$

$v' = x \quad v = \frac{x^2}{2}$

$$\int x \cos x \, dx = \frac{x^2}{2} \cos x - \int \frac{x^2}{2} (-\sin x) \, dx$$

harder than $\int x \cos x \, dx$
Integration by parts

Example \( \int \ln x \, dx \)

Solution

\[ u = \ln x \quad u' = \frac{1}{x} \]
\[ v' = 1 \quad v = x \]

\[ \int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot dx \]
\[ = x \ln x - \int 1 \, dx \]
\[ = x \ln x - x + C \]

\[ \int \ln x \, dx = x \ln x - x + C \]
Integration by parts

Example: compute \( \int_0^{\pi/4} x^2 \sin x \, dx \)

Solution:

\[ u = x^2 \quad \Rightarrow \quad u' = 2x \]
\[ v' = \sin x \quad \Rightarrow \quad v = -\cos x \]

By integration by parts formula,

\[ \int_0^{\pi/4} x^2 \sin x \, dx = \left[ (\frac{\pi}{4})^2 (-\cos (\frac{\pi}{4})) \right] - \int_0^{\pi/4} 2x \cos x \, dx \]

\[ = -\frac{\pi^2}{16} \frac{\sqrt{2}}{2} + 2 \int_0^{\pi/4} x \cos x \, dx \]

Using \( \int x \cos x \, dx = x \sin x + \cos x + C, \)

\[ = -\frac{\pi^2 \sqrt{2}}{32} - \left( 2 \left( \frac{\pi}{4} \sin (\frac{\pi}{4}) + \cos (\frac{\pi}{4}) \right) - \left( 0 \sin 0 + \cos 0 \right) \right) \]

\[ = -\frac{\pi^2 \sqrt{2}}{32} + 2 \left( \frac{\pi}{4} \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 \right) \]

\[ = -\frac{\pi^2 \sqrt{2}}{32} + \frac{\pi \sqrt{2}}{4} + \sqrt{2} - 2. \]
Example: Let $R$ be the region bounded by $y = \ln x$, the $x$-axis, and the line $x = e$. Find the volume of the solid that is generated when the region $R$ is revolved about the $x$-axis.

\[
V = \int_{\frac{1}{4}}^{e} \pi (\ln x - 0)^2 \, dx
\]

(hint: the shell (pancake, disk, slice) is perpendicular to the axis of revolution)

\[
V = \int_{\frac{1}{4}}^{e} \pi \ln^2 x \, dx = \pi \int_{\frac{1}{4}}^{e} \ln^2 x \, dx \quad \Theta
\]

let $u = \ln^2 x$ 
\[
u' = 2\ln x \cdot \frac{1}{x}
\]

\[
V = x
\]

\[
= \pi \left[ e \ln^2 e - \frac{1}{4} \ln^2 1 - \int_{\frac{1}{4}}^{e} x \cdot 2\ln x \cdot \frac{1}{x} \, dx \right]
\]

\[
= \pi \left[ e - 0 - \int_{\frac{1}{4}}^{e} 2 \ln x \, dx \right]
\]

\[
= \pi \left[ e - 2 \int_{\frac{1}{4}}^{e} \ln x \, dx \right]
\]

\[
= \pi \left[ e - 2 \left( e \ln e - e - (1 \ln 1 - 1) \right) \right]
\]

\[
= \pi \left[ e - 2 \left( e - 1 - e - (-1) \right) \right] = \pi e^2 \pi [e - 2]}

Integration by parts. Special trick
(work for $\int e^x \sin x \, dx$ and $\int e^x \cos x \, dx$)

Example Compute $\int e^x \sin(2x) \, dx$

Solution

\[ u = \sin(2x) \quad u' = 2\cos(2x) \]
\[ v' = e^x \quad v = e^x \]

\[ I = \int e^x \sin(2x) \, dx = e^x \sin(2x) - e^0 \sin(2 \cdot 0) - \int 2\cos(2x) e^x \, dx \]
\[ = e\sin 2 - 2 \int e^x \cos(2x) \, dx \quad \Theta \]

\[ u = \cos(2x) \quad u' = -2\sin(2x) \]
\[ v' = e^x \quad v = e^x \]

\[ \Theta \quad e\sin 2 - 2 \left[ \cos(2 \cdot 1) e^1 - \cos(2 \cdot 0) e^0 - \int (-2\sin(2x)) e^x \, dx \right] \]
\[ = e\sin 2 - 2 \left[ e\cos 2 - 1 + 2 \int e^x \sin(2x) \, dx \right] \]

\[ I = e\sin 2 - 2 \left[ e\cos 2 - 1 + 2I \right] \]
\[ I = e\sin 2 - 2e\cos 2 + 2 - 4I \]
\[ 5I = e\sin 2 - 2e\cos 2 + 2 \]

\[ I = \frac{1}{5} \left[ e\sin 2 - 2e\cos 2 + 2 \right] \]
Trigonometric integrals $\int \tan^m x \sec^n x \, dx$

General approach

Is $m$ odd?

- Yes

  - $m$ is odd
    - Use $\tan x = \frac{\sin x}{\cos x}$
    - and $\sec x = \frac{1}{\cos x}$

  - Is $n$ odd?
    - Yes
      - $m$ is even, $n$ is odd
      - Beyond the scope of MATH101
    - No
      - $m$ is even, $n$ is even

  - Is $n \geq 2$?
    - Yes
      - $m$ is even, $n$ is even, $n \geq 2$
        1. Separate one $\sec^2 x$
        2. All other $\sec x \to \tan x$
           using $\sec^2 x = 1 + \tan^2 x$
        3. Substitution
           $u = \tan x$
    - No
      - $m$ is even, $n = 0$
Trigonometric integrals

We study two types of integrals:

\[ \int \sin^m x \cos^n x \, dx \quad \text{and} \quad \int \tan^m x \sec^n x \, dx \]

We are done on Thursday.

We know that for \( \int \tan^m x \sec^n x \, dx \), if \( m \) is odd

use \( \tan x = \frac{\sin x}{\cos x} \) and \( \sec x = \frac{1}{\cos x} \).

Example \( \int \tan x \sec^{2017} x \, dx \)

Solution \( \int \tan x \sec^{2017} x \, dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^{2017} x} \, dx \equiv \)

\[ u = \cos x \quad \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^{2017} x} = f(u)(-\sin x) \]

\[ u' = -\sin x \quad f(u) = -\frac{1}{u} \cdot \frac{1}{u^{2017}} \]

\( \equiv \int -\frac{1}{u} \cdot \frac{1}{u^{2017}} \, du \bigg|_{u = \cos x} \)

\[ = -\int \frac{1}{u^{2018}} \, du \bigg|_{u = \cos x} = -\int u^{-2018} \, du \bigg|_{u = \cos x} \]

\[ = -\left( \frac{1}{-2017} u^{-2017} \right) \bigg|_{u = \cos x} + C \]

\[ = \frac{1}{2017} \cos^{-2017} x + C = \frac{1}{2017} \sec^{2017} x + C \]
Case In \( \int \tan^m x \sec^n x \, dx \), \( m \) is even \( n \) is even, \( n \geq 2 \).
\( n = 2k + 2 \)
\[
\int \tan^m x \sec^{2k+2} x \, dx = \int \tan^m x (\sec^2 x)^k \sec^2 x \, dx
\]
\[
= \int \tan^m x (1 + \tan^2 x)^k \sec^2 x \, dx \tag{1}
\]
let \( u = \tan x \quad u' = \sec^2 x \)
\[
\int u^m (1 + u^2)^k \, du \bigg|_{u = \tan x}
\]
\[
\int \tan^m x \sec^{2k+2} x \, dx = \frac{\tan^{m+2k+2} x}{m+2k+2} + C
\]

Example \[ \int \tan^2 x \sec^2 x \, dx \]
Solution \[ \int \tan^2 x \sec^2 x \, dx = \int \tan^2 x \sec^2 x \, dx \]
\[
= \int u^2 \, du \bigg|_{u = \tan x} = \frac{u^3}{3} \bigg|_{u = \tan x} + C = \frac{\tan^3 x}{3} + C
\]

Example \[ \int \tan^4 x \sec^4 x \, dx \]
Solution \[ \int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dx \]
\[
= \int \tan^4 x (1 + \tan^2 x) \sec^2 x \, dx \]
\[
= \int u^4 (1 + u^2) \, du \bigg|_{u = \tan x} = \int (u^4 + u^6) \, du \bigg|_{u = \tan x}
\]
\[
= \left( \frac{u^5}{5} + \frac{u^7}{7} \right) \bigg|_{u = \tan x} = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C
\]