Theorem (Fundamental theorem of Calculus, p. 45) \(\text{FTC}\) 
Let \(a < b\) and let \(f(x)\) be a continuous function for all \(a \leq x \leq b\).

Part 1 \(\frac{df}{dx} \int_a^x f(t) \, dt \) exists for all \(a \leq x \leq b\).
\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Part 2 Let \(g(x)\) be a continuous function for all \(a \leq x \leq b\). Then let \(g(x)\) be differentiable with \(g'(x) = f(x)\) for all \(a < x < b\). Then,
\[
g(b) - g(a) = \int_a^b f(x) \, dx \quad \text{or equivalently} \quad 
\int_a^b g'(x) \, dx = g(b) - g(a).
\]
Example Compute \( \int_0^1 (1+x^2) \, dx \).

\[
\int_0^1 (1+x^2) \, dx = \int_0^1 dx + \int_0^1 x^2 \, dx = \left. x \right|_0^1 + \frac{1}{3} \left. x^3 \right|_0^1 = (1-0) + \frac{1}{3} (1-0) = \frac{4}{3} \approx 1.333
\]

Example Compute \( \int_{\pi/2}^{5\pi/2} 7 \sin x \, dx \).

\[
\int_{\pi/2}^{5\pi/2} 7 \sin x \, dx = 7 \int_0^{5\pi/2} \sin x \, dx = \left. -7 \cos x \right|_{\pi/2}^{5\pi/2} = -7 \left( -\cos \left( \frac{5\pi}{2} \right) - \cos \left( \frac{\pi}{2} \right) \right) = -7 (0+0) = 0
\]

Example Compute \( \int_0^4 (3e^x + 8 \cos x) \, dx \).

\[
\int_0^4 (3e^x + 8 \cos x) \, dx = \int_0^4 3e^x \, dx + \int_0^4 8 \cos x \, dx = 3 \left. e^x \right|_0^4 + 8 \left. \sin x \right|_0^4 = 3(e^4 - 1) + 8(\sin 4 - 0) = 3(e^4 - 1) + 8 \sin 4
\]

Example Compute the shaded area:

\[
A = 1 - \int_0^{1/2} \frac{3\sqrt{x}}{x} \, dx = 1 - \int_0^{1/2} x^{\frac{1}{3}} \, dx = 1 - \left. \left( \frac{3}{4} x^{\frac{4}{3}} \right) \right|_0^{1/2} = \frac{3}{4} \left( 1^{\frac{4}{3}} - 0^{\frac{4}{3}} \right) = \frac{3}{4}
\]

Guess antiderivative: \( \frac{3}{4} x^{\frac{4}{3} + 1} = \frac{3}{4} x^{\frac{7}{3}} \)

Check: \( \frac{d}{dx} \left( \frac{3}{4} x^{\frac{7}{3}} \right) = \frac{3}{4} \cdot \frac{7}{3} x^{\frac{4}{3}} = x^{\frac{4}{3}} \checkmark \)

Example \( \int_0^4 \frac{3}{x} \, dx = 3 \int_0^4 \frac{1}{x} \, dx = 3 \ln x \bigg|_{x=2}^{x=4} = 3 \ln 4 - 3 \ln 2 = 3 \ln \frac{4}{2} = 3 \ln 2 \).
Example Find the rate of change of \( \int_{3}^{x} (1 + t^{14}) \, dt \).

\[
m = \frac{d}{dx} \int_{3}^{x} (1 + t^{14}) \, dt = 1 + x^{14}
\]

(by the FTC)

Example Find the rate of change of \( \int_{-x}^{x} 2^{t^{14}} \, dt \), \( x > 0 \).

\[
\frac{d}{dx} \int_{-x}^{x} 2^{t^{14}} \, dt = 2^{x^{14}} \cdot 1 - 2^{x^{14}} \cdot (-1) = \frac{x^{14} + 2^{-x^{14}}}{2^{x^{14}} + 2^{-x^{14}}}
\]

Example Find the rate of change of \( I = \int_{0}^{x^2} \sin u \, du \).

Solution Let \( g(x) = \int_{0}^{x^2} \sin u \, du \). \( I = g(x^2) \)

\[
\frac{d}{dx} I = \frac{d}{dx} g(x^2) = g'(x^2) \cdot 2x = [\sin x^2] \cdot 2x
\]

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)
\]

\[
\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) \, dt = \frac{d}{dx} \left[ \int_{0}^{b(x)} f(t) \, dt - \int_{0}^{a(x)} f(t) \, dt \right]
\]

= \frac{d}{dx} \left[ g(b(x)) - g(a(x)) \right]

= g'(b(x)) \cdot b'(x) - g'(a(x)) \cdot a'(x)

= f(b(x)) \cdot b'(x) - f(a(x)) \cdot a'(x)

let \( g(x) = \int_{0}^{x} f(t) \, dt \)
Antiderivatives and the indefinite integrals

Lemma (1.3.8, p. 52) Let \( f(x) \) be a function and let \( F(x) \) be an antiderivative of \( f(x) \). Then,

1. \( F(x) + C \) is also an antiderivative for any constant \( C \).
2. Every antiderivative of \( f(x) \) must be in this form.

Proof 1. \[
\frac{d}{dx} (F(x) + C) = \frac{d}{dx} F(x) + \frac{d}{dx} C = f(x) + 0 = f(x)
\]

2. Let \( F(x) \) and \( G(x) \) be both antiderivatives of \( f(x) \), i.e.,

\[
F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x).
\]

Want \( F(x) - G(x) = C \).

\[
\frac{d}{dx} [F(x) - G(x)] = \frac{d}{dx} F(x) - \frac{d}{dx} G(x) = f(x) - f(x) = 0.
\]

The rate of change of \( F(x) - G(x) \) is 0 everywhere.

\( \Rightarrow \) no change in \( y \)-value.

\( \Rightarrow \) It is a constant! \( F(x) - G(x) = C = \text{const} \)

Definition: The "indefinite integral of \( f(x) \)" is denoted by \( \int f(x) \, dx \) and should be regarded as the general antiderivative of \( f(x) \). In particular, if \( F'(x) = f(x) \), then

\[
\int f(x) \, dx = F(x) + C,
\]

where \( C \) is an arbitrary constant.
Side notes to the table of antiderivatives

\[ \frac{d}{dx} \left[ \frac{a^x}{\ln a} \right] = \frac{d}{dx} \left[ \frac{1}{\ln a} \cdot a^x \right] = \frac{1}{\ln a} \frac{d}{dx} a^x = \frac{1}{\ln a} a^x \ln a = a^x \]

\[ \int \frac{1}{x} \, dx \]

Naive approach: \( \frac{d}{dx} \ln x = \frac{1}{x} \), so
\[ \int \frac{1}{x} \, dx = \ln x + C \]

E.g., \[ \int_{-2}^{1} \frac{1}{x} \, dx = \ln x \bigg|_{x=-2}^{x=1} = \ln (1) - \ln (-2) \]

Problem \( \ln (\text{"negative number"}) \) does not exist.

\[ F(x) = \ln |x| \]

If \( x > 0 \), \quad F(x) = \ln |x| = \ln x \quad F'(x) = \frac{1}{x} \]

If \( x < 0 \), \quad F(x) = \ln |x| = \ln (-x) \quad F'(x) = \frac{1}{-x} (-1) = \frac{1}{x} \]
Table of antiderivatives

1. \( \int 1 \, dx = x + C \)

2. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \) 
   if \( n \neq -1 \)

3. \( \int \frac{1}{x} \, dx = \ln |x| + C \)

4. \( \int e^x \, dx = e^x + C \)

5. \( \int a^x \, dx = \frac{a^x}{\ln a} + C \)

6. \( \int \sin x \, dx = -\cos x + C \)

7. \( \int \sec^2 x \, dx = \tan x + C \)

   Recall: \( \sec x = \frac{1}{\cos x} \)

8. \( \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C = \arcsin x + C \)

9. \( \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C = \arctan x + C \)
Another interpretation for definite integrals

How does the car track the mileage?

Old-style car may not have GPS, so there is no way to know the current location, \( x(t) \). However, it can track how fast the wheels are rotating and conclude the velocity of the car, \( v(t) \).

**Question**: How to recover \( x(t) \) from \( v(t) \)? We know that \( x(0) = 0 \).

\[
\frac{\sigma(\Delta t) \Delta t + \sigma(2\Delta t) \Delta t + \ldots + \sigma(n\Delta t) \Delta t}{\text{velocity}} \approx s(t)
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \sigma(i \Delta t) \Delta t = \text{right Riemann sum } R_n
\]

Say, if we are interested in the mileage in 1h

\[
\Delta t = \frac{1}{n} = \frac{1}{h}
\]

\[
\lim_{n \to \infty} R_n = \int_{0}^{t} \sigma(t) dt = s(t)
\]

Recall the FTC:

\[
\int_{0}^{t} \sigma(u) du = s(t) - s(0)
\]
Example You are in charge of the construction of the flat runway for the planes. You need to know the length of it. The only data related to the matter is a speed of the plane during the take off:

<table>
<thead>
<tr>
<th>time passed after the beginning of the take off, s</th>
<th>0</th>
<th>7</th>
<th>14</th>
<th>21</th>
<th>28</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed, m/s</td>
<td>0</td>
<td>0.6</td>
<td>1.5</td>
<td>2.3</td>
<td>3.2</td>
<td>4</td>
</tr>
</tbody>
</table>

Get the best estimate for the length of the runway.

Numbers are fake. Don’t use these numbers for the runway design -- it is unsafe!

Goal: upper bound on $s(t)$, given values of $\sigma(t)$. $s(t)=\int_0^{35}\sigma(u)\,du$.

$\Delta x=\frac{35}{5}=7$

$R_5=0.6\cdot7+1.5\cdot7+2.3\cdot7+3.2\cdot7+4.7=81.2$

Example(s) Suppose that we observe the particle and we measure its velocity along $x$-axis. We get

$\sigma(t)=\sin(2t)+3\cos\left(\frac{t}{2}\right)$.

Where is the particle located at time $t=2\pi$?

Remark When $t=2\pi$ $\sigma(2\pi)=\sin(4\pi)+3\cos\left(\frac{2\pi}{2}\right)=0+3(-1)=-3<0$.

What does it mean for the particle? For $s(t)$?

Goal $s(3\pi)$? $s(3\pi)-s(0)=\int_0^{3\pi}\sigma(t)\,dt=\int_0^{3\pi}(\sin(2t)+3\cos\left(\frac{t}{2}\right))\,dt$
**Substitution**

Example Recall that \( \frac{d}{dx} \tan x = \sec^2 x \). Compute

\[ \int 5 \sec^2 (5x) \, dx. \]

**Solution** The candidate for the antiderivative:

\( \tan (5x) \)

Check \( \frac{d}{dx} \tan (5x) = \sec^2 (5x) \cdot 5 \) \( \checkmark \)

**Answer** \( \int 5 \sec^2 (5x) \, dx = \tan (5x) + C \)

Observe We used the chain rule

\[ \left( F(u(x)) \right)' = F'(u(x)) \cdot u'(x) \]

\[ \text{In our example} \quad F(x) = \tan x \]

\[ u(x) = 5x \]

\[ f(x) = F'(x) = \sec^2 x \]

\[ u'(x) = 5 \]

Integrate both sides of (*):

\[ \int [F(u(x))]' \, dx = \int F'(u(x)) \cdot u'(x) \, dx \]

Let \( f(x) = F'(x) \).

RHS = \( \int f(u(x)) \cdot u'(x) \, dx \)

LHS = \( F(u(x)) + C \)

\[ \int f(u(x)) \cdot u'(x) \, dx = F(u(x)) + C \]

\[ = \int f(x) \, dx \bigg|_{u=u(x)} + C \]
Substitution (continued)

**Theorem** For any differentiable function $u = u(x)$:

$$
\int f(u(x)) \, u'(x) \, dx = \int f(u) \, du \bigg|_{u=u(x)}
$$

**Example** Consider the integral:

$$
\int \ln^2 x \cdot \frac{1}{x} \, dx = \int \ln u^2(x) \cdot u'(x) \, dx
$$

Guess: $u'(x) = \frac{1}{x}$, $u(x) = \ln x$

$$
\frac{u^3}{3} \bigg|_{u=\ln x} + C
$$

$$
= \ln^3 x + C
$$

**Example** Compute

$$
\int_\pi^{2\pi} \sin^7 x \cdot \cos x \, dx
$$

**Solution** $u'(x) = \cos x$  

$u(x) = \sin x$

$f(u(x)) = \sin^7 x$

$$
\int \sin^7 x \cdot \cos x \, dx = \frac{u^8}{8} \bigg|_{u=\sin x} + C = \frac{\sin^8 x}{8} + C
$$

$$
\int_\pi^{2\pi} \sin^7 x \cdot \cos x \, dx = \frac{\sin^8 (2\pi)}{8} - \frac{\sin^8 \pi}{8} = 0 - 0 = 0
$$