Review of lecture 1

Last time:
- we approximated the area under the curve using approximating rectangles
- we learnt useful notation for sums: Σ notation
- we wanted to approximate the area under the curve with more rectangles.

By the end of the lecture you will be able to:
1. state and compute Riemann sums
2. define the definite integrals and state its relation to the area under the curve
3. list basic properties of the definite integrals and use them to compute some definite integrals
4. state and prove the Fundamental theorem of Calculus (FTC)
5. Use the FTC to compute integrals.
Back to Riemann sums and approximating rectangles.

Let \( f(x) \) be a function defined for all \( a \leq x \leq b \). Then, \( \Delta x \) = width of approximating rectangle = \( \frac{b-a}{n} \).

Riemann sum = \( \sum_{i=1}^{n} f(x_{i}^{*}) \Delta x = \left( \sum_{i=1}^{n} f(x_{i}^{*}) \right) \frac{b-a}{n} \)

where \( x_{i}^{*} \) is any point such that \( x_{i-1} \leq x_{i}^{*} \leq x_{i} \).

Left Riemann sum: all \( y \)-values are taken at leftmost points of the \( k \)-th interval \([x_{i-1}, x_{i}]\).

\[ \sum_{i=1}^{n} f(x_{i-1}) \Delta x = \sum_{i=1}^{n} f(a + (i-1) \Delta x) \frac{b-a}{n} \]

\[ x_{i-1} = a + (i-1) \Delta x \]

Notation \( L_{n} \).

Right Riemann sum: all \( y \)-values are taken at rightmost points of the \( k \)-th interval \([x_{i-1}, x_{i}]\).

\[ \sum_{i=1}^{n} f(x_{i}) \frac{b-a}{n} = \sum_{i=1}^{n} f(a + i \Delta x) \frac{b-a}{n} \]

\[ x_{i} = a + i \Delta x \]

Notation \( R_{n} \).

Midpoint Riemann sum: all if we choose \( y \)-values \( x_{i}^{*} \) to be leftmost

(Left hand end points): \( x_{i}^{*} = a + \frac{(i-1) \Delta x}{2} \).

\[ \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_{i}}{2}\right) \frac{b-a}{n} = \sum_{i=1}^{n} f\left(a + \frac{(i-1) \Delta x}{2}\right) \frac{b-a}{n} \]

\[ x_{i-1} = a + (i-1) \Delta x \]

\[ x_{i} = a + i \Delta x \]

\[ x_{i-1} + x_{i} = a + \frac{(i-1) \Delta x}{2} + \frac{a + (i-1) \Delta x}{2} = a + \frac{2(a + (i-1) \Delta x) - \Delta x}{2} = a + \frac{2i-1}{2} \Delta x = a + (i-\frac{1}{2}) \Delta x \]

Notation \( M_{n} \).
The definite integral

Let \( a, b \) be two real numbers and let \( f(x) \) be a function that is defined for all \( x \) between \( a \) and \( b \). Then we define

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i,n}) \frac{b-a}{n}
\]

when the limit exists. In this case, we say that \( f \) is integrable on the interval from \( a \) to \( b \).

**How to read**

\[
\int_a^b f(x) \, dx = \text{the definite integral of } f(x) \text{ over the interval } a \leq x \leq b
\]

\( f(x) = \text{integrand} \)

\( a, b = \text{limits of integration} \)
The definite integral (continued)

Example Express the area below the curve
\[ f(x) = x^2 \cos^2(2x) + \ln(x) + 1, \quad 2 \leq x \leq 4 \]
as the limit of the left Riemann sum.

Solution \( a = 2, \ b = 4 \), \( \Delta x = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n} \)

\[
A = \int_2^4 f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(2 + (i-1)\frac{2}{n}\right) \frac{2}{n} \\
= \lim_{n \to \infty} \sum_{i=1}^{n} \left( (2 + (i-1)\frac{2}{n})^2 \cos^2\left(2(2 + (i-1)\frac{2}{n})\right) + \ln\left(2 + (i-1)\frac{2}{n}\right) + 1 \right) \frac{2}{n}
\]

Example What area may be found by
\[
A = \lim_{n \to \infty} \sum_{k=1}^{n} (e^{x_k} - e^{-x_k}) \frac{\cos(k/n)}{k+n}
\]

Please correct a typo

Hint: rewrite the expression as the definite integral.

Solution \( \Delta x = \frac{b-a}{n} \), \( a = 0 \), \( b = 1 \)

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}
\]

\[
x_k = \frac{k}{n} = a + \left(k \frac{1}{n}\right) \Delta x = 0 + k \frac{1}{n}
\]

\[
\left[ x_k = a + k \Delta x \right] \quad \text{(right Riemann sum)}
\]

We'd like
\[
\left( e^{x_k} - e^{-x_k} \right) \cos\left(\frac{k}{n}\right)
\]
Pending questions

1. In the learning goals, it is said that we limited ourselves to positive functions. But that was for lecture only. So what happens with \( f_n \) takes \(-\) sign?

2. Are there any guarantees that the limit in the definition of the definite integral exist?

\[
\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k^n) \frac{b-a}{n}
\]

3. Any way to compute the definite integral?

Example. Let \( f(x) = -1 \) for all \( 2 \leq x \leq 3 \).

The area between \( f(x) \) and \( x \)-axis for \( 2 \leq x \leq 3 \) is

\[ A = 1 \]

\[
\int_2^3 f(x) \, dx = \lim_{n \to \infty} \sum_{k=1}^{n} (-1) \frac{3-2}{n} = \lim_{n \to \infty} \sum_{k=1}^{n} (-1) \frac{1}{n} = \lim_{n \to \infty} \left[ (-1) \sum_{k=1}^{n} \frac{1}{n} \right]
\]

\[ = \lim_{n \to \infty} \left[ (-1) \frac{n}{n} \right] = \lim_{n \to \infty} (-1) = -1 \]

Remark \( \lim_{n \to \infty} \sum_{k=1}^{n} g_n(k) \neq \sum_{k=1}^{n} \lim_{n \to \infty} g_n(k) \)

The definite integral = "signed" area in a sense that we count area above \( x \)-axis with \( +\), below \( x \)-axis with \( -\).
Group work

1. Compute \( \int_{-1}^{1} \text{sign}(x) \, dx \), where \( \text{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases} \).

Hint: If you see a function with cases, split the integral, so only one case will hold for each integral.

\[ \int_{-1}^{1} \text{sign}(x) \, dx = \int_{-1}^{0} \text{sign}(x) \, dx + \int_{0}^{1} \text{sign}(x) \, dx \]

\[ \int_{-1}^{1} \text{sign}(x) \, dx = -\frac{1}{3} + 1 = 0 \]

2. Compute \( \int_{2}^{5} (1 + 2x) \, dx \).

Hint: Sketch it.

\[ f(x) = 1 + 2x \]

\[ f(2) = 5 \]

\[ f(3) = 7 \]

\[ A = \frac{5 + 7}{2} \cdot 1 = 6; \quad A = 5.1 + \frac{1}{2} \cdot 1.2 = 6 \]

3. Compute \( \int_{-1}^{2} (3x - 2) \, dx \).

Hint: Sketch it.

\[ f(x) = 3x - 2 \]

\[ f(-1) = -3 - 2 = -5 \]

\[ f(2) = 4 \]

\[ 3x - 2 = 0 \quad x = \frac{2}{3} \]

4. Compute \( \int_{-2}^{2} (x^5 + 4x) \, dx \).

\[ \int_{0}^{1} (1-x^2) \, dx \]

\[ y = \sqrt{1-x^2} \quad 0 \leq x \leq 1 \]

\[ A = \frac{\pi}{4} \]
Properties of the definite integrals

1. \[ \int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx \]

2. \[ \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]
   \[ \int_a^b (f(x) - g(x)) \, dx = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \]

Exercise. Justify it.

Hint. Use the definition involving Riemann sums. Then use that \( \sum_{k=1}^n a_k + b_k = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \) and show that

\[ \lim_{n \to \infty} (F(n) + G(n)) = \lim_{n \to \infty} F(n) + \lim_{n \to \infty} G(n) \]

3. \[ \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx \]

   E.g. \( \int_{-1}^{2} (2\cos x - \sin x) \, dx = 2\int_{-1}^{2} \cos x \, dx - \int_{-1}^{2} \sin x \, dx \)

4. If \( f(x) \geq 0 \) for all \( a \leq x \leq b \), then \( \int_a^b f(x) \, dx \geq 0 \)

If \( m \leq f(x) \leq M \) for all \( a \leq x \leq b \), then \( \int_a^b f(x) \, dx \leq M(b-a) \)

\[ \int_a^b f(x) \, dx \geq m(b-a) \]
Properties of the definite integrals

5. If \( f(x) \) is increasing over the interval \( a \leq x \leq b \), then

\[
L_n \leq \int_a^b f(x) \, dx \leq R_n
\]

If \( f(x) \) is decreasing over the interval \( a \leq x \leq b \), then

\[
L_n \geq \int_a^b f(x) \, dx \geq R_n
\]

6. (Symmetry)

If \( f(x) \) is an odd function over \(-a \leq x \leq a\),

\[
\int_{-a}^{a} f(x) \, dx = 0
\]

If \( f(x) \) is an even function over \(-a \leq x \leq a\),

\[
\int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f(x) \, dx
\]

Example: Compute \( \int_{-1}^{1} \sin^5(\pi x) \, dx \)

Idea: \( f(x) = \sin^5(\pi x) \) is odd.

Let's check: \( f(-x) = \sin^5(-\pi x) \)

\[
= (-\sin(\pi x))^5
= (-1)^5 \sin^5(\pi x)
= -1 \cdot \sin^5(\pi x)
= -\sin^5(\pi x)
= -f(x) \Rightarrow \text{odd}
\]

\[
\int_{-1}^{1} f(x) \, dx = 0.
\]
Theorem: Let \( f(x) \) be a function on the interval \( a \leq x \leq b \). If

(i) \( f(x) \) is continuous on \( a \leq x \leq b \), or
(ii) \( f(x) \) has a finite number of jump discontinuities on \( [a,b] \) (and is otherwise continuous),
then \( f(x) \) is integrable on \( [a,b] \).

There will be no proof or justification of this theorem in MATH 101. You need to know the statement of the theorem for your final (and, potentially, quiz).
Fundamental theorem of calculus

Let's consider \( g(x) = \int_0^x f(t)\,dt \). Let's find \( g'(x) \).

Recall the definition from Calculus I:

\[
g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \left( \frac{x + \Delta x}{x} \right) \left( \frac{\int_0^{x+\Delta x} f(t)\,dt - \int_0^x f(t)\,dt}{\Delta x} \right)
\]

\[
= \lim_{\Delta x \to 0} \left( \frac{x + \Delta x}{x} \right) \frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x}
\]

\[
= \lim_{\Delta x \to 0} \left( \frac{x + \Delta x}{x} \right) f(x)
\]

\[
= f(x)
\]

1. \( \frac{d}{dx} \int_0^x f(t)\,dt = f(x) \)

2. \( g(x) = \int_0^x g'(t)\,dt \). Let \( x = b \). \( g(0) = 0 \)

\[
g(b) - g(a) = \int_a^b g'(t)\,dt
\]

\[
g(b) = f(b)
\]
Theorem (Fundamental theorem of calculus, p. 45)

Let \( a < b \) and let \( f(x) \) be a continuous function for all \( a \leq x \leq b \).

Part 1

\[
\frac{d}{dx} \int_a^x f(t) \, dt \text{ exists for all } a \leq x \leq b.
\]

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Part 2

Let \( g(x) \) be a continuous function for all \( a \leq x \leq b \). Then let \( g(x) \) be differentiable with \( g'(x) = f(x) \) for all \( a < x < b \). Then,

\[
g(b) - g(a) = \int_a^b f(x) \, dx \text{ or equivalently}
\]

\[
\int_a^b g'(x) \, dx = g(b) - g(a).
\]
Example Compute \( \int_0^1 (1 + x^2) \, dx \).

\[ \int_0^1 (1 + x^2) \, dx = \]

Example Compute \( \int_{\pi/2}^{5\pi/2} 7\sin x \, dx = \)

Example Compute \( \int_0^4 (3e^x + 8\cos x) \, dx \).

Example Compute the shaded area.