Review session. Topic: series
August 8th, 2017

Remark. I will leave the answers in the calculator-ready form.


Problem 1. Find the value for the following series.

1. \[
\sum_{n=1}^{\infty} (\ln n - \ln(n+1)) = \lim_{N \to \infty} \sum_{n=1}^{N} (\ln n - \ln(n+1)) = \lim_{N \to \infty} \left( (\ln 1 - \ln(2)) + (\ln 2 - \ln(3)) + (\ln 3 - \ln(4)) + \ldots + (\ln N - \ln(N+1)) \right)
\]
\[
= \lim_{N \to \infty} \left( \ln 1 - \ln(N+1) \right) = -\infty, \text{ the series diverges to } -\infty.
\]

2. \[
\sum_{n=4}^{\infty} \frac{3 \cdot (-1)^n + 2 \cdot 2^n + 1 \cdot (-3)^n}{4^n} = 3 \sum_{n=4}^{\infty} \frac{(-1)^n}{4^n} + 2 \sum_{n=4}^{\infty} \frac{2^n}{4^n} + \sum_{n=4}^{\infty} \frac{(-3)^n}{4^n}
\]
\[
= 3 \frac{\frac{1}{4^4}}{1 - \frac{1}{4}} + 2 \frac{\frac{2}{4^4}}{1 - \frac{2}{4}} + \frac{\frac{-3}{4^4}}{1 - \frac{-3}{4}}
\]


Problem 2. For the following series, determine if the series converges or diverges. Justify your answer.

1. \[
\sum_{n=1}^{\infty} \frac{n^2 + n - 1}{n^2 + 2n + 3}
\]

Step 0. Take a brief look if the divergence test is applicable.

\[
\lim_{n \to \infty} \frac{n^2 + n - 1}{n^2 + 2n + 3} = \lim_{n \to \infty} \frac{n^2 \left( -1 + \frac{1}{n} - \frac{1}{n^2} \right)}{n^2 \left( 1 + \frac{2}{n} + \frac{3}{n^2} \right)} = -1 \neq 0,
\]
by the divergence test, the series diverges.

2. \[
\sum_{n=1}^{\infty} \frac{n^{3/2} + n^{-11/2}}{n^{3/2} + (n^{11/2})} \sim \sum_{n=1}^{\infty} \frac{n^{3/2}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \quad n \times \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}, \quad \text{p-series } 1/p = 3/2, \text{ converges b/c it's a p-series.}
\]

Special case: sum like polynomial over polynomial, here: ignore if powers of \( n \) are not integers.
Then use the limit comparison test.

By the limit comparison test,

\[
\lim_{n \to \infty} \frac{n^{3/2} + n^{-11/2}}{n^{3/2} + n^{-11/2}} = \lim_{n \to \infty} \left( \frac{n^{3/2} + n^{-11/2}}{n^{3/2}} \right) = \lim_{n \to \infty} \left( \frac{n^{3/2} + n^{-11/2}}{n^{3/2}} \right) = \lim_{n \to \infty} 1,
\]
the series \( \sum_{n=1}^{\infty} \frac{n^{3/2} + n^{-11/2}}{n^{3/2}} \) converges.
3. \[ \sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n^3} \], where \(c_n\) is the \(n^{th}\) digit of \(e = 2.718\ldots\). Special case: contains expressions whose limit does not exist when \(n \to \infty\) (even not \(\infty\) or \(-\infty\)). Here it is \(n^{th}\) digit of the number \(e\). The only test that does not ask to compute the limit is the comparison test.

We would like to apply the comparison test. Since \(c_n\) is a digit, its value is one of \(0, 1, 2, 3, 4, 5, 6, 7, 8, 9\). Therefore, \(|c_n| \leq 9\). Finally,

\[ \left| \frac{(-1)^n \frac{c_n}{n^3}}{n^3} \right| = \frac{9}{n^3} \leq 9. \]

Note that \(\sum_{n=1}^{\infty} \frac{9}{n^3} = 9 \sum_{n=1}^{\infty} \frac{1}{n^3}\) converges b/c its \(p\)-series \(\frac{1}{n^3}\) converges. By the comparison test, \(\sum_{n=1}^{\infty} \frac{(-1)^n c_n}{n^3}\) converges.

4. \[ \sum_{n=1}^{\infty} \frac{n^{100}}{n^4} \]. Special case: contains factorials. Use the ratio test.

By the ratio test, \(\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} (a_n)^{-1} \right| = \lim_{n \to \infty} \left( \frac{n+1}{n+1} \right) \left( \frac{n+1}{n+1} \right) = \lim_{n \to \infty} \left( \frac{n+1}{n+1} \right) \left( \frac{n+1}{n+1} \right) = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 0, \]

so the series converges.

5. \[ \sum_{n=1}^{\infty} \left( \tan \left( \frac{1}{n} \right) \right)^n \]. Special case: complex expression taken to power \(n\), \(-n\), \(n^2\), etc. Use root test.

In other words, use the root test, when it is easy to take the \(n^{th}\) root of the term.

By the root test, \(\lim_{n \to \infty} \sqrt[n]{\left( \tan \left( \frac{1}{n} \right) \right)^n} = \lim_{n \to \infty} \left| \tan \left( \frac{1}{n} \right) \right| = \left| \tan \left( \frac{1}{n} \right) \right| = \left| \tan \left( \frac{1}{n} \right) \right| = 0, \]

the series converges.

6. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n^{ev}}} \]. Special case: all terms are positive and function \(\frac{\sqrt[n]{x}}{\sqrt[n]{x}}\) is easy to integrate. Use the integral test.

Let \(f(x) = \frac{1}{\sqrt[n]{x}}\). Note that \((i)\ f(x) = 0 \ (ii)\ f(x)\ is\ continuous\ (iii)\ f(x)\ is\ decreasing, \ x \geq 1\). \[ \int_{1}^{\infty} \frac{1}{\sqrt[n]{x}} \ dx = \lim_{R \to \infty} \left[ -2 e^{-\frac{1}{\sqrt[n]{x}}} \right]_{1}^{R} = \lim_{R \to \infty} \left( -2 e^{-\frac{1}{\sqrt[n]{R}}} - (-2 e^{-\frac{1}{\sqrt[n]{1}}}) \right) = 2 e^{-1} \]
7. \[ \sum_{n=1}^{\infty} \cos(n\pi) \sqrt{n} \]
\[ a_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}} \quad \cos(n\pi) = (-1)^n \]

Special case: alternating series. If it is easy to check conditions of the alternating series test, use it. Otherwise try other tests.

(i) \[ a_n > 0 \] \checkmark
(ii) \[ \{a_n\} \text{ is decreasing} \] \checkmark
(iii) \[ \lim_{n \to \infty} a_n = 0 \] \checkmark

By the alternating series test, the series converges.

8. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3n)!}{2^n(n!)} \]

It is alternating series test, but it's hard to check conditions. Try the ratio test.

\[ \lim_{n \to \infty} \frac{\frac{(-1)^n(3(n+1))!}{2^{n+1}(n+1)!}}{\frac{(-1)^{n-1}(3n)!}{2^n(n-1)(n)!}} = \lim_{n \to \infty} \frac{3(n+3)!}{n!(n+1)!} \frac{2^n}{2^n} = \lim_{n \to \infty} \frac{(3n+3)!}{(n+1)!} \frac{1}{2} \]

By the ratio test, the series diverges.

Part 3. The error bound.

For the integral test and the alternating series test, we can bound the truncation error \[ \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \].

Problem 3. We know that \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \ln n}} \] converges. How many terms do we need to add such that the truncation error is below \( 2 \cdot 10^{-10} \)? We will use that \( \int e^{-\sqrt{x}}/\sqrt{x} \, dx = -2e^{-\sqrt{x}} + C \) without justification.

Solution. We know that if the integral test conditions are met for \( f(x) \) such that \( a_n = f(n) \),

\[ \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_{N}^{\infty} f(x) \, dx \]

\[ \int_{N}^{\infty} f(x) \, dx = \int_{N}^{\infty} \frac{1}{\sqrt{x}e^{\sqrt{x}}} \, dx = \lim_{R \to \infty} \left( -2e^{-\sqrt{R}} - (-2e^{-\sqrt{N}}) \right) = 2e^{-\sqrt{N}} \]

Want: \( 2e^{-\sqrt{N}} \leq 2 \cdot 10^{-10} \), \( e^{-\sqrt{N}} \leq 10^{-10} \), \( -\sqrt{N} \leq -10 \), \( \sqrt{N} \geq 10 \), \( N \geq 100 \).

\[ N = 100 \]

Problem 4. We know that \[ \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} \] converges. Establish the error bound for

\[ \sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} - \sum_{n=1}^{100} \frac{\cos(n\pi)}{\sqrt{n}} \leq \left| \frac{\cos(100\pi)}{\sqrt{1001}} \right| = \frac{1}{\sqrt{1001}} \]

(we checked the conditions of the alternating series test before)
Part 4. Absolute and conditional convergence.

Each series is either divergent, either conditionally convergent, either absolutely convergent.

Problem 5. Determine if the following series is divergent, conditionally convergent or absolutely convergent.

1. \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{\sqrt{n}}}{\sqrt{n}} \).

   Trick. It's easy to check the convergence of \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{\sqrt{n}}}{\sqrt{n}} \).

   We showed that \( \sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} \) converges, so \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\sqrt{n}}}{\sqrt{n}} \) converges absolutely, and therefore, the series \( \sum_{n=1}^{\infty} \frac{(-1)^n e^{-\sqrt{n}}}{\sqrt{n}} \) converges.

2. \( \sum_{n=1}^{\infty} \frac{(-1)^n n^6 - n^4 + n^3 - n^2 - n + 1}{n^{10} + n^8 + n^6 + n^4 + n^2 + 1} \).

   We start with checking absolute convergence.

   \[ \sum_{n=1}^{\infty} \left| \frac{n^5 - n^4 + n^3 - n^2 - n + 1}{n^9 + n^8 + n^6 + n^4 + n^2 + 1} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5} \]

   \( \frac{1}{n^5} \) converges because it's a p-series with \( p = 5 \).

   By the limit comparison test,

   \[ \lim_{n \to \infty} \frac{n^5 - n^4 + n^3 - n^2 - n + 1}{n^9 + n^8 + n^6 + n^4 + n^2 + 1} = \lim_{n \to \infty} \frac{n^5 - n^4 + n^3 - n^2 - n + 1}{n^9 + n^8 + n^6 + n^4 + n^2 + 1} \cdot \frac{n^5}{n^5} \]

   \[ = \lim_{n \to \infty} \left( \frac{n^5}{n^9} \right) \]

   \[ = \lim_{n \to \infty} \left( \frac{1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} - \frac{1}{n^4} + \frac{1}{n^5}}{1 + \frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6} + \frac{1}{n^8} + \frac{1}{n^{10}}} \right) \]

   \( \lim_{n \to \infty} \) cancel circled terms

   Therefore, by the limit comparison test, the series \( \sum_{n=1}^{\infty} \left| \frac{n^5 - n^4 + n^3 - n^2 - n + 1}{n^9 + n^8 + n^6 + n^4 + n^2 + 1} \right| \) converges. Therefore,

   \( \sum_{n=1}^{\infty} \frac{(-1)^n n^5 - n^4 + n^3 - n^2 - n + 1}{n^{10} + n^8 + n^6 + n^4 + n^2 + 1} \) converges absolutely and so the series converges.