The alternating series test (ctd)

**Theorem** Let \( \{a_n\}_{n=1}^{\infty} \) be the sequence such that

(i) \( a_n \geq 0 \) for all \( n \)

(ii) \( \{a_n\}_{n=1}^{\infty} \) is decreasing, i.e., \( a_n \geq a_{n+1} \) for all \( n \)

(iii) \( \lim_{n \to \infty} a_n = 0 \).

Then, \( a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n \) converges.

Moreover, the value of the series \( \sum_{n=1}^{\infty} (-1)^{n-1} a_n \) is always between \( S_N \) and \( S_{N+1} \).

**Example** \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \)

**Remark** \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges.

**Solution** \( a_n = \frac{1}{n} \)

\( \circled{\text{ (i) } a_n \geq 0} \)

\( \circled{\text{ (ii) } \{a_n\}_{n=1}^{\infty} \text{ is decreasing}} \)

\( \circled{\text{ (iii) } \lim_{n \to \infty} \frac{1}{n} = 0} \)

By the alternating series test, \( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \) converges.

**Example** Let's consider the series \( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \), where \( n! = 1 \cdot 2 \cdot 3 \cdot 4 \ldots \cdot n \) (\( n! \) is read as \( n \) factorial).

**Solution**
Theorem Suppose that \( \{a_n\}_{n=1}^{\infty} \) satisfies the conditions of the alternating series test. Then, \( \sum_{n=1}^{\infty} (-1)^{n-1}a_n \) converges, and let \( S = \sum_{n=1}^{N} (-1)^{n-1}a_n \). Then, the truncation error \( |S - \sum_{n=1}^{\infty} (-1)^{n-1}a_n| \) is between 0 and \( a_{N+1} \).

Remark This statement will NOT be stated on quiz nor final exam. You need to memorize it by yourself.

Example Later in this course, we will show that 
\[
e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}, \text{ where } n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n.
\]

(a) How far is \( S_4 = 1 - 1 + \frac{1}{2} - \frac{1}{6} \) to \( e^{-1} \)?

Solution

(b) How many terms do we need to approximate \( e^{-1} \) to within 0.001?
Example We know that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges, so we want to approximate it by its $N$th partial sum $\sum_{n=1}^{N} \frac{(-1)^n}{n}$. What $N$ should be such that the truncation error

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{N} \frac{(-1)^n}{n} \right|$$

is less than 0.001?