Example Compute \( \int_0^1 (1+x^2) \, dx \).

\( \int_0^1 (1+x^2) \, dx = \)

Example Compute \( \int_{\pi/2}^{5\pi/2} 7\sin x \, dx = \)

Example Compute \( \int_0^4 (3e^x + 8\cos x) \, dx \)

Example Compute the shaded area:

Example Compute \( \int_{1/2}^4 \frac{3}{x} \, dx \)
Example Find the rate of change of \( \int_3^x (1 + t^{14}) dt \).

Example Find the rate of change of \( \int_{-x}^x \frac{1}{2} 2^{t+1} dt \).

Example Find the rate of change of \( \int_0^x \sin k u du \).

\[
\frac{d}{dx} \left[ \int_{a(x)}^{b(x)} f(t) dt \right] =
\]
Another interpretation for definite integrals

How does the car track the mileage?

Old-style car may not have GPS, so there is no way to know the current location, \( x(t) \). However, it can track how fast the wheels are rotating and conclude the velocity of the car, \( v(t) \).

Question: How to recover \( x(t) \) from \( v(t) \)? We know that \( x(0) = 0 \).
You are in charge of the construction of the flat runway for the planes. You need to know the length of it. The only data related to the matter is a speed of the plane during the take off:

<table>
<thead>
<tr>
<th>time passed after the beginning of the take off, s</th>
<th>0</th>
<th>7</th>
<th>14</th>
<th>21</th>
<th>28</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>speed, m/s</td>
<td>0</td>
<td>0.6</td>
<td>1.5</td>
<td>2.3</td>
<td>3.2</td>
<td>4</td>
</tr>
</tbody>
</table>

Get the best estimate for the length of the runway.

*Numbers are fake. Don’t use these numbers for the runaway design -- it is unsafe!

**Examples** Suppose that we observe the particle and we measure its velocity along x-axis. We get

\[ v(t) = \sin(2t) + 3 \cos(\frac{t}{2}) \]

Where is the particle located at time \( t = 3\pi \)?

Remark when \( t = 2\pi \) \( v(2\pi) = \sin(4\pi) + 3\cos(\frac{2\pi}{2}) = 0 + 3(-1) = -3 < 0 \).

What does it mean for the particle? For \( s(t) \)?
Antiderivatives and the indefinite integrals

Lemma (1.3.8, p. 52) Let \( f(x) \) be a function and let \( F(x) \) be an antiderivative of \( f(x) \). Then,

1. \( F(x) + C \) is also an antiderivative for any constant \( C \).
2. Every antiderivative of \( f(x) \) must be in this form.

Proof

2. Let \( F(x) \) and \( G(x) \) be both antiderivatives of \( f(x) \), i.e.,
   \[
   F'(x) = f(x) \quad \text{and} \quad G'(x) = f(x).
   \]

Definition The "indefinite integral of \( f(x) \)" is denoted by \( \int f(x) \, dx \) and should be regarded as the general antiderivative of \( f(x) \). In particular, if \( F'(x) = f(x) \), then

\[
\int f(x) \, dx = F(x) + C,
\]
where \( C \) is an arbitrary constant.
Table of antiderivatives

\[ \int 1 \, dx = x + C \]

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad \text{if} \ n \neq -1 \]

Why? Remember that \( \frac{d}{dx}(\ln x) = \frac{1}{x} \).

\[ \int \frac{1}{x} \, dx \]

\[ \int e^x \, dx = e^x + C \]

\[ \int a^x \, dx = \frac{a^x}{\ln a} + C \]

\[ \int \sin x \, dx = -\cos x + C \]

\[ \int \cos x \, dx = \sin x + C \]

\[ \int \sec^2 x \, dx = \tan x + C \]

Recall: \( \sec x = \frac{1}{\cos x} \)

\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \]

\[ \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C \]
Side notes to the table of antiderivatives
Example Recall that \( \frac{d}{dx} \tan x = \sec^2 x \). Compute \( \int 5 \sec^2 (5x) \, dx \).

Solution The candidate for the antiderivative:

Check

Answer

Observe We used the chain rule

\[
[F(u(x))]' = F'(u(x))u'(x)
\]

In our example \( F(x) = \), \( f(x) = F'(x) = \), \( u(x) = \), \( u'(x) = \)

Integrate both sides of (\*):

\[
\int [F(u(x))]' \, dx = \int F'(u(x)) u'(x) \, dx
\]

Let \( f(x) = F'(x) \).

RHS =

LHS =
Substitution (continued)

**Theorem** For any differentiable function $u = u(x)$:

$$\int f(u(x)) \, u'(x) \, dx = \int f(u) \, du \bigg|_{u = u(x)}$$

**Example** Consider the integral:

$$\int \ln^2 x \cdot \frac{1}{x} \, dx$$

**Example** Compute

$$\int_{\pi}^{2\pi} \sin^3 x \cos x \, dx.$$
Theorem. For any differentiable function $u(x)$:

$$
\int_a^b f(u(x)) u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du
$$
Example: \[ \int_{1}^{2} \frac{1}{x^2} \frac{1}{x} \, dx \]

Example: \[ \int \frac{9}{(7-9x)^5} \, dx \]

Example: \[ \int_{0}^{2\ln 2} \frac{e^x + 1}{e^x} \, dx \]

Example: \[ \int_{1}^{10} \frac{10^x}{10^x + 10} \, dx \]
Example \[ \int_0^1 (2x + 1) \sin(x^2 + x) \, dx \]

Example \[ \int_0^2 x \sqrt{16 - x^4} \, dx \]