Areas and distances

Example. You've bought a real estate near the ocean, and you want to double-check its area for the tax purposes. However, the area is not regularly shaped. Ideas?

For our example

Potential issues

Table of values

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1/6</th>
<th>1/3</th>
<th>1/2</th>
<th>2/3</th>
<th>5/6</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + x^2</td>
<td>1</td>
<td>1.03</td>
<td>1.11</td>
<td>1.25</td>
<td>1.44</td>
<td>1.69</td>
<td>2</td>
</tr>
</tbody>
</table>

Use the table of values to approximate the areas in (a) - (c)

(a) \( A \approx \)

(b) \( A \approx \)

(c) \( A \approx \)

For simplicity, we assume that all approximating rectangles have the same width.
Observation: More and thinner rectangles =>

Let's consider 100 terms! ... How to write them down...

\[ \sum^\Sigma \text{-notation} \]

Let \( f(x) \geq 0 \) for all \( 0 \leq x \leq 1 \) and let \( f \) be a continuous function. The area under the curve may be approximated by 8 rectangles: (use rightmost points only).

Issue: If you show to a friend or colleague

Alternative \( \sum \text{ notation} \)

Notation: \( a_m + a_{m+1} + a_{m+2} + \cdots + a_n = \sum_{k=m}^{n} a_k \)

Examples:

1. \( \sum_{k=3}^{5} k = \)
2. \( \sum_{i=2}^{6} (-1)^i = \)
3. \( \sum_{k=1}^{10} (n+k) = \)

You may find the following formula useful:

\[ \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \]
4. \( 5^2 + 6^2 + \cdots + 10^2 = \)
5. \( 2 + 2^{\frac{1}{2}} + 2^{\frac{1}{3}} + \cdots + 2^{\frac{1}{4}} = \)
Properties of $\Sigma$-notation

(same as properties of sum)

1. $\sum_{k=n}^{m} (a_k + b_k) = \sum_{k=n}^{m} a_k + \sum_{k=n}^{m} b_k$
   
   Why?

2. $\sum_{k=n}^{m} ca_k = c \sum_{k=n}^{m} a_k$

Example: Compute $\sum_{k=2}^{1000} (3 + 7k)$. You may use $\sum_{k=1}^{m} k = \frac{k(k+1)}{2}$

Solution: $\sum_{k=2}^{1000} (3 + 7k) =$

Remark: Index of summation is not a real variable. To avoid confusion, do not write it outside of sum.

// iClicker question
Back to Riemann sums and approximating rectangles.

Let \( f(x) \) be a function defined for all \( a \leq x \leq b \). Then,

\[
\Delta x = \text{width of approximating rectangle} = x_i - x_{i-1}
\]

**Riemann sum**

\[
\sum (x_i^r)
\]

**Left right Riemann sum**: all \( y \)-values are taken at right most points of the \( k \)th interval \([x_{i-1}, x_i]\)

**Left Riemann sum**: all if we choose \( y = x_i^l \) to be leftmost (left hand end point)

**Midpoint Riemann sum**: on each subinterval, we take the midpoint
The definite integral

Let $a, b$ be two real numbers and let $f(x)$ be a function that is defined for all $x$ between $a$ and $b$. Then we define

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_{i,n}) \frac{b-a}{n}$$

when the limit exists. In this case, we say that $f$ is integrable on the interval from $a$ to $b$.

How to read

$$\int_a^b f(x) \, dx =$$

$f(x) =$

$a, b =$