Short answer question

1. 3 marks iClicker question.

Only one of the following statements is always true; determine which one is true.

A. If \( a_n \to 0 \) as \( n \to \infty \), then the series \( \sum_{n=1}^{\infty} a_n \) converges.
B. If sequence \( \{a_n\}_{n=1}^{\infty} \) is decreasing, then it converges.
C. The sequence \( \left\{ a_n = \frac{n^3 + 1}{n^3 - 1} \right\}_{n=2}^{\infty} \) is divergent.
D. If series \( \sum_{n=1}^{\infty} a_n \) converges, then the sequence \( \{S_N\}_{N=1}^{\infty} \), where \( S_N = \sum_{n=1}^{N} a_n \) is \( N^{th} \) partial sum of the series \( \sum_{n=1}^{\infty} a_n \), also converges.
E. The series \( \sum_{n=1}^{\infty} 2^n \) converges.

Answer: D

Solution: D holds from the definition of series. We call the series converges if \( \lim_{n \to \infty} S_N \) exists, i.e., \( \{S_N\} \) is convergent.

Why other statements do not hold (you do not have to do all of them on final, this part is for your own understanding).

A. \( a_n = 1/n \) converges to 0 as \( n \to \infty \), but \( \sum_{n=1}^{\infty} 1/n \) diverges.
B. \( a_n = -n \) is a decreasing sequence but it diverges to \( -\infty \).
C. \( \lim_{n \to \infty} \frac{n^3 + 1}{n^3 - 1} = \lim_{n \to \infty} \frac{n^3}{n^3} = 1 \).
E. By the divergence test, \( \lim_{n \to \infty} 2^n = \infty \neq 0 \), so the series diverge. Alternatively, you may notice that it is geometric series with \( r = 2 \).
2. Each part is worth 3 marks. Please write your answers in the boxes if provided. If not, put a box around your final answer. Note that you must include the correct accompanying work to receive full credit.

(a) If the following series converges, compute its sum. If it diverges, please specify whether it diverges to infinity, negative infinity or neither.

\[ \sum_{n=1}^{\infty} \left( e^{-n^2} - e^{-(n+1)^2} \right). \]

Answer: \( e^{-1} \).

Solution: First, let’s consider \( N \)th partial sum of the series.

\[
S_N = \sum_{n=1}^{N} \left( e^{-n^2} - e^{-(n+1)^2} \right)
= (e^{-1} - e^{-2^2}) + (e^{-2^2} + e^{-3^2}) + \ldots + (e^{-N^2} - e^{-(N+1)^2}) = e^{-1} - e^{-(N+1)^2}
\]

In fact, this series is a telescoping series. Finally,

\[
\sum_{n=1}^{\infty} \left( e^{-n^2} - e^{-(n+1)^2} \right) = \lim_{N \to \infty} S_N = \lim_{N \to \infty} \left( e^{-1} - e^{-(N+1)^2} \right) = e^{-1}.
\]

Marking scheme:

• 1 mark for considering \( N \)th partial sum
• 1 mark for computing \( N \)th partial sum and canceling out most of the terms
• 1 mark for computing the limit of partial sums.

If you just expanded the series and canceled out the terms, you will get 2 marks out of 3.

(b) If the following series converges, compute its sum (and simplify your answer). If it diverges, please specify whether it diverges to infinity, negative infinity or neither.

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^{2n+1}}{9^n}.
\]

Answer: 17/10
Solution: Note that

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^{2n+1}}{9^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{9^n} + \sum_{n=1}^{\infty} \frac{2 \cdot 4^n}{9^n}. \]

The first series in the RHS is a geometric series with \( a = \frac{1}{9} \) and \( r = -\frac{1}{9} \) and the second series in the RHS is a geometric series with \( a = \frac{2 \cdot 4}{9} = \frac{8}{9} \) and \( r = \frac{4}{9} \).

Therefore,

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 2^{2n+1}}{9^n} = \frac{1/9}{1 - (-1/9)} + \frac{8/9}{1 - 4/9} = \frac{1}{9 + 1} + \frac{8}{9 - 4} = \frac{1}{10} + \frac{8}{5} = \frac{1 + 16}{10} = \frac{17}{10} \]

Marking scheme:

- 1 mark for splitting one series into two series and recognizing that each of them is geometric series.
- 2 marks for computing each series and simplifying the value of each series. (1 mark for each series)
(c) Consider the region \( R \) which is bounded by \( x^2 + y^2 = 4 \), \( x^2 + y^2 = 1 \), \( x = 0 \) and \( y = 0 \) such that \( x \geq 0 \) and \( y \geq 0 \) (see the sketch below). This region may be considered as a quarter-ring. The area of \( R \) is equal \( 3\pi/4 \) (You do NOT need to show this fact.) Write down the expression for x-coordinate of the centroid of the region A. Do not evaluate any integrals; simply write down the expression asked for.

**Solution:** The x-coordinate of the centroid may be found

\[
\bar{x} = \frac{\int_0^1 x(\sqrt{4 - x^2} - \sqrt{1 - x^2}) \, dx + \int_1^2 x\sqrt{4 - x^2} \, dx}{3\pi/4}
\]

or

\[
\bar{x} = \frac{\int_0^2 x\sqrt{4 - x^2} \, dx - \int_0^1 x\sqrt{1 - x^2} \, dx}{3\pi/4}
\]

**Marking scheme:**

- 1 mark for considering \( \sqrt{1 - x^2} \) and \( \sqrt{4 - x^2} \)
- 1 mark for integrating \( x(\sqrt{4 - x^2} - \sqrt{1 - x^2}) \).
- 1 mark for the final answer

If you write the area using the integral, that’s also fine (if the integral, or more precisely, some of integrals is correct)

(d) Does the following series converge or diverge? Justify your answer.

\[
\sum_{n=5}^{\infty} \frac{2^{1/n}n^2}{(2n + 1)^2}
\]

**Solution:** Let us apply the divergence test, i.e., let us find \( \lim_{n \to \infty} \frac{2^{1/n}n^2}{(2n + 1)^2} \). We want to rewrite the limit as a product of limits:

\[
\lim_{n \to \infty} \frac{2^{1/n}n^2}{(2n + 1)^2} = \lim_{n \to \infty} 2^{1/n} \lim_{n \to \infty} \frac{n^2}{(2n + 1)^2}
\]

Wednesday, August 2nd
provided at least one of the limits is a finite number.

Let’s compute the first limit. Note that since \( g(x) = 2^x \) is continuous,

\[
\lim_{n \to \infty} 2^{1/n} = 2^{\lim_{n \to \infty} 1/n} = 2^0 = 1.
\]

The second limit is taken of the ratio of polynomials, degrees of numerator and denominator are 2.

\[
\lim_{n \to \infty} \frac{n^2}{(2n + 1)^2} = \lim_{n \to \infty} \frac{n^2}{n^2 (2 + 1/n)^2} = \frac{1}{4}.
\]

Finally, the original limit is

\[
\lim_{n \to \infty} \frac{2^{1/n} n^2}{(2n + 1)^2} 2^{n+1/n} = \lim_{n \to \infty} 2^{1/n} \lim_{n \to \infty} \frac{n^2}{(2n + 1)^2} = 1 \cdot \frac{1}{4} = \frac{1}{4} \neq 0.
\]

Since the limit of the correspondence sequence is not zero, by the divergence test, series is divergent.

**Marking scheme:**

- 1 mark for referring to the divergence test or applying it (even if incorrectly)
- 1 mark for computing the limit
- 1 mark for applying the divergent test

Potential errors: if you tried to use the integral test, then you can get up to two marks. 1 mark is assigned to verifying the conditions of the integral test, 1 mark for considering the correspondent improper integral.
(e) The cylindrical tank shown below, with a radius of 2 m and height 3 m, is full of water by 1/3, i.e., the water depth is 1 m. Using the fact that the weight of water is 1000 \( kg/m^3 \), set up the integral that represents the work (in joules) required to pump the water out of the outlet pipe 1 m above the top of the cylindrical tank. Do NOT evaluate the integral.

![Cylindrical tank diagram]

**Solution:** Consider the slice of width \( \Delta y \), its mass is \( \Delta m = 1000\pi r^2 \Delta y = 4000\pi \Delta y \). If this slice is located \( y \) meters from the bottom of the cylinder, this slice should be lifted by \( 4 - y \). The work required to pump this slice is

\[
\Delta W = g(4 - y) \Delta m = g(4 - y) \cdot 4000\pi \Delta y.
\]

Finally, the total work done

\[
W = \int_0^1 g(4 - y) \cdot 4000\pi \, dy.
\]

**Marking scheme:**

- 1 mark for having term \( 4 - y \) in the integral
- 1 mark for having \( 4000\pi = \pi r^2 \cdot 1000 \) in the integral
- 1 mark for correct limits of integration
Long answer question—you must show your work

3. **5 marks** Consider the separable differential equation

\[ \frac{dy}{dx} = 44y^2 x^{10}. \]

(a) **3 marks** Solve the differential equation. (You need to find ALL solutions.)

**Solution:** This differential equation (DE) is a separable DE. Therefore, if \( y^2 \neq 0 \),

\[ \int \frac{1}{y^2} \, dy = \int 44 x^{10} \, dx. \]  

(1)

Computing the indefinite integrals from the LHS and the RHS, we get

\[ -\frac{1}{y} = 4x^{11} + C. \]

Therefore,

\[ y = -\frac{1}{4x^{11} + C}, \]  

where \( C \) is any constant.

Finally, consider the case \( y^2 = 0 \). Then \( \frac{dy}{dx} = 44y^2 x^{10} = 0 \) that means that the solution is constant, i.e., \( y = c \) for some \( c \). Solving \( y^2 = 0 \) gives us \( y = 0 \). The final answer is

\[ y = -\frac{1}{4x^{11} + C}, \]  

where \( C \) is any constant and \( y = 0 \).

**Marking scheme:**

- 1 mark for getting the expression (1) (see the solution).
- 1 mark for getting the solution \( y = -\frac{1}{4x^{11} + C} \).
- 1 mark for the solution \( y = 0 \).

(b) **2 marks** Find an equation of the curve that satisfies the differential equation above and whose \( y \)-intercept is 1. Feel free to use the result of part (a).

**Solution:** Note that \( y = 0 \) is not the curve we are looking for. Therefore, we are looking for the curve

\[ y = -\frac{1}{4x^{11} + C}, \]

where \( y(0) = 1 \). We get the equation

\[ 1 = -\frac{1}{0 + C}. \]
which implies $C = -1$.

Therefore, the curve in question is

$$y = -\frac{1}{4x^{11} - 1}.$$  

**Marking scheme:**

- 1 mark for plugging $(x, y) = (0, 1)$ into the formula obtained in part (a)
- 1 mark for finding the value of $C$ and writing down the final answer.
Long answer question—you must show your work

4. 7 marks Consider the series

\[ \sum_{n=1}^{\infty} \frac{n}{e^n}. \]

We want to use the integral test to determine the convergence of this series.

(a) 1 mark Prove that

\[ \lim_{R \to \infty} -\frac{(R + 1)}{e^R} = 0. \]

(We will use this result in next parts of this question.) Hint. Use the L'Hospital rule.

**Solution:** Let us investigate the limits of the numerator and denominator as \( R \to \infty. \)

\[ \lim_{R \to \infty} -\frac{(R + 1)}{e^R} = \left[ -\infty \atop \infty \right] \]

We use the L’Hospital rule to evaluate the limit.

\[ \lim_{R \to \infty} -\frac{(R + 1)}{e^R} = \lim_{R \to \infty} -\frac{1}{e^R} = 0. \]

(b) 2 marks We want to apply the integral test to prove that \( \sum_{n=1}^{\infty} \frac{n}{e^n} \) converges. In this part, verify the conditions of the integral test.

**Solution:** Let \( f(x) = \frac{x}{e^x} = xe^{-x} \) for \( x \geq 1. \) Clearly, \( f(x) \geq 0 \) and \( f(x) \) is continuous for \( x \geq 1. \) We still need to verify that \( f(x) \) is decreasing \( x \geq 1. \) From differential calculus, we know that function is decreasing if and only if \( f'(x) \leq 0. \)

\[ f'(x) = e^{-x} - xe^{-x} = (1 - x)e^{-x}. \]

For \( x \geq 1, 1 - x \leq 0 \) and \( e^{-x} > 0. \) Therefore, \( f'(x) \leq 0 \) which implies that \( f(x) \) is decreasing for \( x \geq 1. \)

**Marking scheme:**

- 1 mark for stating \( f(x) \geq 0 \) and \( f(x) \) is continuous.
- 1 mark for checking \( f \) is decreasing as \( x \) increases.
(c) **1 mark** Apply the integral test to the series above and prove that series is convergent.

Note that in this part of the question, if you determine the convergence of the series using other methods, you will not get any marks.

You may use the result in part (a) and the following formula without justification.

\[ \int \frac{x}{e^x} \, dx = -\frac{x+1}{e^x} + C. \]

**Solution:** We want to use the integral test, and its conditions are verified in part (b). The integral test says that the series \( \sum_{n=1}^{\infty} \frac{n}{e^n} \) converges if and only if \( \int_{1}^{\infty} \frac{x}{e^x} \, dx \) converges.

\[
\int_{1}^{\infty} \frac{x}{e^x} \, dx = \lim_{R \to \infty} \int_{1}^{R} \frac{x}{e^x} \, dx = \lim_{R \to \infty} \left( -\frac{x+1}{e^x} \right)_{x=1}^{x=R} \\
= \lim_{R \to \infty} \left( -\frac{R+1}{e^R} + \frac{2}{e} \right) = \left( -\lim_{R \to \infty} \frac{R+1}{e^R} \right) + 2e^{-1} = 2e^{-1}.
\]

We have shown that the integral converges. Therefore, according to the integral test, series also converges.

(d) **3 marks** Let \( S = \sum_{n=1}^{\infty} \frac{n}{e^n} \) and \( f(x) = \frac{x}{e^x} \). The Integral Test Theorem claims that under certain conditions for function \( f(x) \) which we verified in part (b),

\[ |S - S_N| \leq \int_{N}^{\infty} f(x) \, dx. \]

What is the smallest value of \( N \) for which the \( N \)th partial sum \( S_N \) of the series is at most \( 2018/e^{2017} \) away from \( S \)? Justify your answer.

You may use the result in part (a) and the following formula without justification.

\[ \int \frac{x}{e^x} \, dx = -\frac{x+1}{e^x} + C. \]

**Answer:** \( N = 2017. \)

**Solution:** According to the Integral Test Theorem,

\[ |S - S_N| \leq \int_{N}^{\infty} \frac{x}{e^x} \, dx. \]

We want \( |S - S_N| \leq 2018/e^{2017} \). Therefore, we need to find \( N \) such that

\[ \int_{N}^{\infty} \frac{x}{e^x} \, dx \leq 2018/e^{2017}. \]
Using the result of part (a), we can compute the improper integral.

\[
\int_{N}^{\infty} \frac{x}{e^x} \, dx = \lim_{R \to \infty} \int_{N}^{R} \frac{x}{e^x} \, dx = \lim_{R \to \infty} \left( \frac{-R}{e^R} - \frac{-N}{e^N} \right) = \frac{N + 1}{e^N}.
\]

We need to find the smallest \(N\) such that

\[
\frac{N + 1}{e^N} \leq \frac{2018}{e^{2017}}.
\]

Clearly, \(N = 2017\).

**Marking scheme:**

- 1 mark for stating that we want the improper integral to be equal to the requested error bound (if you say that we want the improper integral to be less or equal to the requested error bound)
- 1 mark for computing the indefinite integral (using limit)
- 1 mark for concluding \(N = 2017\).