Short answer question

1. 1 mark  iClicker question. Consider the definite integral \( \int_{\pi/6}^{\pi/3} (\tan x)^{4000} (\sec x)^{10000} \, dx \). What substitution will you use to evaluate the integral? Do NOT evaluate the integral. Only the answer in the box will be marked.

A. \( u = \tan x \)
B. \( u = \sec x \)
C. \( u = \sin x \)
D. \( u = \cos x \)
E. the correct substitution is not listed above.

Answer: A

Solution: This integral is of the type \( \int_a^b \tan^m x \sec^n x \, dx \), where \( m = 4000 \) is even and \( n = 10000 \) is even. According to the rule stated in the lecture, we do substitution \( u = \tan x \).
2. 12 marks Each part is worth 3 marks. Please write your answers in the boxes. You must show your work.

(a) Compute \( \int (x^2 - 2x) \ln x \, dx \). Remember that every indefinite integral must include an additive constant, which you can enter as \(+C\). Please show your work.

\[
\text{Answer: } \left( \frac{x^3}{3} - x^2 \right) \ln x - \frac{1}{9} x^3 + \frac{1}{2} x^2 + C
\]

Solution: We will use the integration by parts to evaluate this integral. Let \( u = \ln x \) and \( v' = x^2 - 2x \). Then, \( u' = \frac{1}{x} \) and \( v = \frac{x^3}{3} - x^2 \). Using the integration by parts,

\[
\int (x^2 - 2x) \ln x \, dx = \left( \frac{x^3}{3} - x^2 \right) \ln x - \int \frac{1}{x} \left( \frac{x^3}{3} - x^2 \right) \, dx
\]

\[
= \left( \frac{x^3}{3} - x^2 \right) \ln x - \left( \frac{1}{3} \int x^2 \, dx - \int x \, dx \right)
\]

\[
= \left( \frac{x^3}{3} - x^2 \right) \ln x - \frac{1}{9} x^3 + \frac{1}{2} x^2 + C
\]

Marking scheme:

- 1 point for letting \( u = \ln x \) and \( v' = x^2 - 2x \).
- 1 point for (correct) applying the integration by parts
- 1 point for the final answer (including \(+C\))

(b) Compute \( \int \sin^{17} x \cos^3 x \, dx \). Remember that every indefinite integral must include an additive constant, which you can enter as \(+C\). Please show your work.

\[
\text{Answer: } \frac{\sin^{18} x}{18} - \frac{\sin^{20} x}{20} + C
\]

Solution: This integral is of the type \( \int \sin^m x \cos^n x \, dx \), where \( n \) is odd. (you may use the technique when \( m \) is odd, but then solution is harder)

\[
\int \sin^{17} x \cos^3 x \, dx = \int \sin^{17} x \cos^2 x \cos x \, dx = \int \sin^{17} x (1 - \sin^2 x) \cos x \, dx.
\]

Let \( u = \sin x \), then \( u' = \cos x \). Using the substitution rule,
\[
\int \sin^{17} x (1 - \sin^2 x) \cos x \, dx = \int u^{17} (1 - u^2) \, du \\
= \int (u^{17} - u^{19}) \, du \\
= \left( \frac{u^{18}}{18} - \frac{u^{20}}{20} \right) \bigg|_{u = \sin x} + C \\
= \frac{\sin^{18} x}{18} - \frac{\sin^{20} x}{20} + C
\]

**Marking scheme:**

- 1 marks for considering \( u = \sin x \) or \( u = \cos x \).
- 1 mark for performing the substitution correctly (the chosen one) If you forget to write \( \bigg|_{u = \sin x} \), you still get this mark.
- 1 mark for back substitution \( u = \sin x \) and final answer.
(c) Consider the region \( R \) bounded by \( y = x, \ x = 0, \) and \( y = 1. \) Sketch the region. Find the volume of the solid obtained by revolving \( R \) about the \( y \)-axis. Please simplify your answer.

**Solution:** We start from sketching the region.

Here the axis of revolution is vertical, so the slice is horizontal. We want to use the washer method to compute the volume of the solid of revolution. The limits of integrals are 0 and 1 (see sketch.) The length of each disk is \( x = y. \)

\[
V = \int_{0}^{1} \pi y^2 dy = \pi \left[ \frac{y^3}{3} \right]_{y=0}^{y=1} = \frac{1}{3} \pi.
\]

**Marking scheme:**

- 1 mark for sketch
- 1 mark for setting up the integral with respect to variable \( y, \) i.e., of the form \( V = \int_{a}^{b} f(y)dy \) with some function \( f(y). \)
- 1 mark for the final answer.

Alternative solution: note that the obtained solid of the revolution is a cone with radius 1 and height 1, so its volume is \( \frac{1}{3}. \) You get full marks for this solution.

If you consider the wrong region, you lose 1 mark that is assigned to the sketch.

(d) Compute the definite integral \( \int_{-1}^{e} (2x + 3)e^x \, dx. \)

**Answer:** \( 2e^{e+1} + e^e + e^{-1} \)
Solution: We will use the integration by parts to evaluate the integral. Let $u = 2x + 3$ and $v' = e^x$. Then, $u' = 2$ and $v = e^x$. By integration by parts,
\[
\int_{-1}^{e} (2x + 3)e^x \, dx = (2e + 3)e^e - (2(-1) + 3)e^{-1} - \int_{-1}^{e} 2e^x \, dx
\]
\[
= 2e^{e+1} + 3e^e - e^{-1} - 2e^x \bigg|_{x=-1}^{x=e}
\]
\[
= 2e^{e+1} + 3e^e - e^{-1} - (2e^e - 2e^{-1})
\]
\[
= 2e^{e+1} + e^e + e^{-1}
\]

Marking scheme:

- 1 mark for identifying that we need to do the integration by parts. If you just perform the integration by parts as the first step (even for wrong choice of $u$ and $v$), you still get this point.
- 1 mark for correct implementation of the integration by parts for sensible choice of $u$ and $v$.
- 1 mark for the final answer.
Long answer question—you must show your work

3. **8 marks** Let \( R \) be the region bounded by \( y = x^2 - 1 \), \( x = 0 \), and \( y = 0 \) which lies to the right of the y-axis. Sketch the region. Set up the integral that represents the volume of the solid obtained by rotation \( R \) about the \( y = -1 \). Please simplify the answer.

**Answer:** \( \frac{4}{5} \pi \)

**Solution:** First, let us sketch the region. Note that the region is revolved about the horizontal axis, so we have vertical slices.

![Diagram of the region R]

We want to use the washer method to set up the definite integral whose value is a volume \( V \) of the solid of revolution. Note that the lower limit of integration is 0 (see sketch), the upper one is 1.

Therefore,

\[
V = \int_0^1 \pi \left[ (R(x) - (-1))^2 - (r(x) - (-1))^2 \right] \, dx,
\]

where \( R(x) = 0 \) and \( r(x) = x^2 - 1 \).

Let us set up explicit expression for \( V \).

\[
V = \int_0^1 \pi \left[ (0 + 1)^2 - (x^2 - 1 + 1)^2 \right] \, dx = \int_0^1 \pi (1^2 - x^4) \, dx
\]

\[
= \pi \left[ \int_0^1 1 \, dx - \int_0^1 x^4 \, dx \right] = \pi \left[ x - \frac{x^5}{5} \right]_0^1
\]

\[
= \pi \left[ 1 - \frac{1}{5} \right] - \left[ 0 - \frac{0}{5} \right] = \frac{4}{5} \pi.
\]
Marking scheme:

- 1 mark for correct sketch.
- 1 mark if the integral is with respect to $x$ or if you draw a vertical slice.
- 1 mark for the limits of the integral
- 2 marks for figuring out that $R(x) = 0$ bounds the region from above and $r(x) = x^2 - 1$ bounds the region from below. If you don’t state explicitly what the curves are, but you have used them in the definite integral in the correct order, you still get 2 marks. You get 1 mark for correct identifying two curves ($y = 0$ and $y = x^2 - 1$) but assuming that $R(x) = x^2 - 1$ and $r(x) = 0$.
- 1 mark for subtracting -1 from $R(x)$ and $r(x)$, i.e., using $(R(x) + 1)^2$ and $(r(x) + 1)^2$ as parts of the integrand
- 1 mark if the definite integral represented volume is set up correctly (no missing $\pi$, missing squares, etc.)
- 1 mark for correct evaluation of the integral and the final answer.

If you considered the wrong region, the marking scheme is still applicable, but you lose 2 more marks.
4. [9 marks] Part (a) is worth 3 marks, part (b) is worth 6 marks.

(a) Compute \( \int_0^{\pi/6} \sin^2 \theta d\theta \). Please simplify your answer.

Answer: \( \frac{\pi}{12} - \frac{\sqrt{3}}{8} \)

Solution: This is the definite integral of the type \( \int_a^b \sin^m x \cos^n x \, dx \), where \( m = 2 \) and \( n = 0 \). We will use \( \sin^2 x = \frac{1 - \cos(2x)}{2} \).

\[
\int_0^{\pi/6} \sin^2 \theta d\theta = \int_0^{\pi/6} \frac{1 - \cos(2\theta)}{2} d\theta.
\]

Now let \( u = 2\theta \). Then, \( u' = 2 \). When \( \theta = 0 \), then \( u = 0 \). When \( \theta = \frac{\pi}{6} \), then \( u = \frac{\pi}{3} \).

When \( \theta = 0 \), then \( u = 0 \). Using the substitution rule,

\[
\int_0^{\pi/6} \frac{1 - \cos(2\theta)}{2} d\theta = \int_0^{\pi/3} \frac{1 - \cos u}{2} \cdot \frac{1}{2} du
\]

\[
= \frac{1}{4} (u - \sin u) \bigg|_0^{\pi/3}
\]

\[
= \frac{1}{4} \left( \left( \frac{\pi}{3} - \sin \left( \frac{\pi}{3} \right) \right) - (0 - \sin 0) \right)
\]

\[
= \frac{1}{4} \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right)
\]

\[
= \frac{\pi}{12} - \frac{\sqrt{3}}{8}
\]

Marking scheme:

- 1 mark for the formula for \( \sin^2 x \).
- 1 mark for computation of the definite or indefinite integral of \( \cos 2\theta \).
- 1 mark for simplification.

Remark 1. \( \frac{2\pi - 3\sqrt{3}}{24} \) is also a simplified version of the final answer, so you get the full marks if your answer is in this form (provided your solution is correct).

Remark 2. You do not have to do the substitution if you can guess the antiderivative of \( \cos(2\theta) \) and using the Fundamental Theorem of Calculus, compute the definite integral. Marks will not be deducted if you have done question this way.
(b) Compute $\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} \, dx$. You may use the value of the definite integral $\int_0^{\pi/6} \sin^2 \theta \, d\theta$ obtained in part (a).

**Answer:** $\frac{\pi}{12} - \frac{\sqrt{3}}{8}$

**Solution:** $\sqrt{1-x^2}$ hints that we should try the trigonometric substitution $x = \sin \theta$. Then, $x' = \cos \theta$. When $x = 0$, $\theta = \arcsin(0) = 0$. When $x = -\frac{1}{2}$, $\theta = \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Then,

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = |\cos \theta|.$$

For $0 \leq \theta \leq \frac{\pi}{6}$, the values of $\cos \theta$ are positive, so

$$|\cos \theta| = \cos \theta.$$

Finally, by the substitution rule,

$$\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}} \, dx = \int_0^{\pi/6} \frac{\sin^2 \theta}{\cos \theta} \cos \theta \, d\theta = \int_0^{\pi/6} \sin^2 \theta \, d\theta.$$

Finally, we computed the definite integral $\int_0^{\pi/6} \sin^2 \theta \, d\theta$ in part (a).

**Marking scheme:**

- 1 mark for letting $u = \sin x$ (or $u = \cos x$ which I do NOT recommend but it will do)
- 2 mark for the change of limits of integration if ultimately values of $\arcsin 0$ and $\arcsin(-\frac{1}{2})$ are computed. 1 mark if the limits of integration are changed according to the substitution rule, but $\arcsin 0$ and $\arcsin(-\frac{1}{2})$ are NOT computed.
- 1 mark for determining the sign of $\cos \theta$ and using it to simplify $|\cos \theta|$.
- 1 mark for the implementation of substitution rule
- 1 mark for reducing the integral to the integral considered in part (a)