1. Let $\mathcal{X}$ be a Banach space and let $T : \mathcal{X} \to \mathcal{X}^*$ be a linear map (not necessarily bounded) satisfying

$$(T(x))(y) = (T(y))(x)$$

for all $x, y \in \mathcal{X}$. Show that $T$ is a bounded operator. (Hint: Use the uniform boundedness principle).

2. Let $\alpha = (\alpha_n)$ be a given sequence of real numbers. Assume that $\sum_n |\alpha_n| |x_n| < \infty$ for every element $x = (x_n)$ in $\ell^3$. Prove that $\alpha \in \ell^{3/2}$. (Hint: Use the uniform boundedness principle on an appropriate sequence of finite-rank operators. Also note that the converse of the above statement follows from Hölder inequality).

3. Suppose that $(X, \mu)$ is a measure space such that $\mu(X) < \infty$. Let $T : L^2(\mu) \to L^2(\mu)$ be a bounded operator. Suppose that the range of $T$ is contained in $L^3(\mu)$. Show that $T$ is bounded as an operator from $L^2(\mu)$ into $L^3(\mu)$. (Hint: Use the closed graph theorem).

4. Let $\mathcal{X}$ and let $\mathcal{Y}$ be normed vector spaces and let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Recall that the adjoint $T^* \in \mathcal{L}(\mathcal{Y}^*, \mathcal{X}^*)$ is defined by $T^* f = f \circ T$ for all $f \in \mathcal{Y}^*$. Show that $T^*$ is injective if and only if the range of $T$ is dense in $\mathcal{Y}$.

5. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. In the class, we saw that every finite-rank operator is compact. In this exercise, we prove a partial converse. If $T$ is a compact operator and the range of $T$ is closed in $\mathcal{Y}$, show that $T$ is a finite-rank operator. (Hint: Use the open mapping theorem).

6. Let $\mathcal{X} = \ell^p$ with $1 \leq p \leq \infty$. Let $(\lambda_n)$ be a bounded sequence in $\mathbb{R}^n$. Consider the operator $\mathcal{L}(\mathcal{X}, \mathcal{X})$ defined by

$$Tx = (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n, \ldots),$$
where
\[ x = (x_1, x_2, \ldots, x_n, \ldots). \]

Show that \( T \) is a compact operator from \( \mathcal{X} \) into \( \mathcal{X} \) if and only if \( \lambda_n \to 0 \).