1. Assume that $X$ is a standard $(\mathcal{F}_t)$-Poisson process with rate $\lambda > 0$.
   (a) Prove that $t \mapsto X_t$ is increasing $\mathbb{Z}_+$-values a.s.
   (b) Prove that $\Delta X(t) = 0$ or 1 for all $t > 0$ a.s.
   (c) If $S_n = \inf \{ t \geq 0 : X_t = n \}$, show that $\tau_n = S_n - S_{n-1}$ ($n \geq 1$) is a sequence of iid exponential random variables with mean $\lambda^{-1}$. Note that $X_t = \sum_{n=1}^{\infty} 1_{\{S_n \geq t\}}$, which would give a simpler way of constructing a Poisson process.
   (d) Verify directly that $M_t = X_t - \lambda t$, $M_t^2 - X_t$, $M_t^2 - \lambda t$ are $(\mathcal{F}_t)$-martingales.

2. Show that there is a constant $c_4 > 0$ such that for any martingale $\{M_n : n \in \mathbb{Z}_+\}$, if $d_n = M_n - M_{n-1}$ and $[M] = \sum_{n=1}^{\infty} d_n^2$, then $\mathbb{E}([M]^2) \leq c_4 \mathbb{E}((M^*)^4)$.

3. Let $M \in \mathcal{M}_{0,\text{loc}}$.
   (a) If $M_t^* \in L^1$ for all $t > 0$, then prove that $M$ is a martingale.
   (b) Assume that $M$ is continuous (but not the hypothesis in (a)).
   (i) Prove that if $\mathbb{E}([M]^2) < \infty$ for all $t > 0$, then $M$ is a square integrable $(\mathcal{F}_t)$-martingale and $M_t^2 - [M]_t$ is an $(\mathcal{F}_t)$-martingale.
   (ii) Prove that for any $a, b > 0$, $P(M_t^* \geq a, [M]_t \leq b) \leq b/a^2$. (Hint: One approach is to consider $M^T$, where $T = \inf \{ t \geq 0 : [M]_t \geq b \}$.)