MATH 421/510
HW 5

Due: April 5, 2016 (in class)

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1. Chapter 6, Exercise 11 in Folland.
2. Let $(X, \mathcal{M}, \mu)$ be a measure space, where μ is a semifinite measure (recall the definition from page 25 in Folland). Let $g \in L^\infty(X, \mu)$.
   (a) Compute the operator norm of the 'multiplication operator' $M_g : L^2(X, \mu) \to L^2(X, \mu)$ defined by $M_g f = fg$.
   (b) Show that the spectrum of M_g is the essential range of g.
   (c) As shown in the class, using Riesz-Fréchet representation theorem one might consider the adjoint operator M_g^* as an operator in $L^2(X, \mu)$. Show that $M_g^* = M_h$ for some $h \in L^\infty(X, \mu)$. Compute h. When is M_g self-adjoint? (Hint: It might be useful to remember that all the functions involved above are *complex* valued).
3. Let $T \in \mathcal{L}(X, X)$ where X is a Banach space.
   (a) If λ is in the residual spectrum of T, then show that λ is in the point spectrum of $T^* \in \mathcal{L}(X^*, X^*)$. (Hint: You have done a similar question in the previous homework).
   (b) If λ is in the point spectrum of T^*, then show that λ is either in the point spectrum or the residual spectrum of T.
   (c) Deduce that if T is a self-adjoint operator in a Hilbert space, then the residual spectrum of T is empty.
4. In question 2, consider the special case, $X = [0, 1]$, $g(x) = x$ and μ is the Lebesgue measure. Compute the spectrum of M_g and classify each point in the spectrum of M_g into one of the following categories: point spectrum, continuous spectrum and residual spectrum.
5. (Spectrum of the shift operators) Let $X = \ell^2(\mathbb{N})$ denote the space of square summable *real-valued* sequences. Define the right shift $S_r \in \mathcal{L}(X, X)$ as
$$S_r((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots),$$
and left shift S_l as
$$S_l((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots).$$

(a) Compute the adjoint operators S_l^* and S_r^* as elements of $\mathcal{L}(X, X)$ (using the canonical identification given by Riesz-Fréchet representation theorem).

(b) Prove that the point spectrum of S_r is empty.

(c) Show that the spectrum of S_r is $[-1, 1]$. (Hint: What is $\|S_r\|$? Is $\lambda u - S_r u = (1, 0, 0, \ldots)$ solvable when $\lambda \in [-1, 1]$ and $u \in \ell^2$?)

(d) Prove that the point spectrum of S_l is $(-1, 1)$. Compute the corresponding eigenspaces.

(e) Compute the residual spectrum of S_r and the residual spectrum of S_l. (Hint 1: See question 3, Hint 2: ℓ^2 is reflexive.)

(f) Compute the continuous spectrum of S_r and the continuous spectrum of S_l.

(g) Is S_l a compact operator? Is S_r a compact operator?

Remark. The significance of question 2 is that the following: Every bounded, self-adjoint operator on a Hilbert space is essentially a 'multiplication operator'. This is one of the many "spectral theorems" in functional analysis. A precise formulation of the above statement can be found in Theorem 3.22 of Leonard Gross' lecture notes. The above spectral theorem can be interpreted as the generalization of the following familiar result in linear algebra: Every hermitian matrix is diagonalizable.

Unfortunately, many applications involve operators that are self-adjoint but unbounded (for instance, the energy operator in quantum mechanics, the Laplacian operator in differential equations, the generator of symmetric Markov processes in probability theory). Therefore there is a need to study the spectral theory of unbounded self-adjoint operators. It is one of the topics that is beyond the scope of this course but nevertheless important for diverse applications.